THE BEHAVIOUR OF THE ELLIPTIC MODULAR FUNCTION AND NTH DERIVATIVE OF ELLIPTIC MODULAR FUNCTION CAN BE EXPRESSED IN TERMS OF DIFFERENTIAL OPERATOR (Δ).

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Introduction

The absolute invariant J(z), of the modular group M arises in the theory of elliptic functions, (Where the variable is usually denoted by J.). Elliptic modular functions and related functions play an important role in the theory of numbers for some application see Hardy (1940) the absolute invariant, J(z) has the property that J(α) is an integral algebraic number where α has a positive imaginary part and the root of a quadratic equation with integer coefficients. The algebraic equations with integer coefficients satisfied by certain J (α) are the So-Called class equations for imaginary quadratic number- fields see Fricke (1928), Fueter (1924, 1927): see also Schneider (1936), Hecke (1939). A new and for reaching development was originated by Hecke (1935, 1937, 1939, 1940 a, 1940b). See also Peterson (1939) and for certain numerical results, Zassenhaus (1941).

For some results which are relevant for the subject of this section, although they appear as special cases of a much more general theory, see Siegel(1935). Apostol T.M., Modular Functions and Dirichlet Series in Number Theory.

2 - Formulation:

$$D = \frac{d}{dr}$$

Δ = forward difference

h = increment in the interval

$f_1(\tau) = \lambda(\tau)$

$f_2 = 1-f_1$

$$f_3 = \frac{1}{f_1}$$

$$f_4 = \frac{1}{1-f_1}$$

$$f_5 = \frac{f_1}{f_1-1}$$

$$f_6 = \frac{f_1-1}{f_1}$$

2.1 Theorem : For $f_1(\tau) = \lambda(\tau)$, to prove that

$$D^n f_1 = \frac{1}{h^n} \Delta^n f_1$$

Proof:

$$\Delta f_1 = hDf_1$$

$$\Delta^2 f_1 = h^2 D^2 f_1$$

$$\Delta^3 f_1 = h^3 D^3 f_1$$

Similarly

$$\Delta^n f_1 = h^n D^n f_1$$

$$D^n f_1 = \frac{1}{h^n} \Delta^n f_1$$

2.2 Theorem : For $f_2(\tau) = 1-f_1(\tau)$, to prove that $D^n f_2 = \frac{\Delta^n f_1}{h^n}$

Proof:

$$f_2 = 1-f_1$$

$$Df_2 = -\frac{\Delta f_1}{h}$$

$$D^2 f_2 = -\frac{\Delta^2 f_1}{h^2}$$

Similarly

$D^n f_2 = \frac{\Delta^n f_1}{h^n}$

2.3 Theorem: For $f_3(\tau) = \frac{1}{f_1(\tau)}$, prove that $D^n f_3 = -\frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{f_1^2}\right)$

$f_3 = \frac{1}{f_1}$

$Df_3 = -\frac{1}{h}\frac{\Delta f_1}{f_1^2}$

$D^2 f_3 = -\frac{1}{h^2}\Delta\left(\frac{\Delta f_1}{f_1^2}\right)$

$D^3 f_3 = -\frac{1}{h^3}\Delta^2\left(\frac{\Delta f_1}{f_1^2}\right)$

Similarly

$D^n f_3(\tau) = -\frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{f_1^2}\right)$

2.4 Theorem: For $f_4(\tau) = \frac{1}{1-f_1(\tau)}$ To prove that $D^n f_4(\tau) = \frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$

Proof : $f_4(\tau) = \frac{1}{1-f_1(\tau)}$

$Df_4(\tau) = \frac{1}{h}\frac{\Delta f_1}{(1-f_1)^2}$

$D^2 f_4(\tau) = \frac{1}{h^2}\Delta\left(\frac{\Delta f_1}{(-f_1+1)^2}\right)$

$D^3 f_4(\tau) = \frac{1}{h^3}\Delta^2\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$

Similarly

$$D^n f_4(\tau) = \frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$$

2.5 Theorem: For $f_5(\tau) = \frac{f_1(\tau)}{f_1(\tau)-1}$, prove that $D^n f_5(\tau) = \frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$

Proof:

$$f_5(\tau) = \frac{f_1}{f_1-1}$$

$$Df_5 = -\frac{1}{h}\frac{\Delta f_1}{(f_1-1)^2}$$

$$D^2 f_5 = -\frac{1}{h^2}\Delta\left(\frac{\Delta f_1}{(f_1-1)^2}\right)$$

Similarly

$$D^n f_5(\tau) = \frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$$

2.6 Theorem: For

$f_6(\tau) = \frac{f_1(\tau)-1}{f_1(\tau)}$, prove that $D^n f_6(\tau) = \frac{1}{h^n}\Delta^{n-1}\left(\frac{\Delta f_1}{{f_1}^2}\right)$

Proof:

$$f_6(\tau) = \frac{f_1(\tau)-1}{f_1(\tau)}$$

$$Df_6(\tau) = -\frac{1}{h}\frac{\Delta f_1}{(f_1)^2}$$

$$D^2 f_6(\tau) = -\frac{1}{h^2}\,\Delta\left(\frac{\Delta f_1}{{f_1}^2}\right)$$

$$D^3 f_6(\tau) = \frac{1}{h^3}\,\Delta^2\left(\frac{\Delta f_1}{{f_1}^2}\right)$$

Similarly

$$D^n f_6(\tau) = \frac{1}{h^n}\,\Delta^{n-1}\left(\frac{\Delta f_1}{{f_1}^2}\right)$$

2.7 Corollary: In above results if h = 1 the results change in the following form.

$$D^n f_1 = \Delta^n f_1$$

$$D^n f_2 = -\Delta^n f_1$$

$$D^n f_3 = -\Delta^{n-1}\left(\frac{\Delta f_1}{{f_1}^2}\right)$$

$$D^n f_4 = \Delta^{n-1}\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$$

$$D^n f_5 = -\Delta^{n-1}\left(\frac{\Delta f_1}{(1-f_1)^2}\right)$$

$$D^n f_6 = \Delta^{n-1}\left(\frac{\Delta f_1}{{f_1}^2}\right)$$

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