2014

ISSN 2278-5655

# Aarhat Multidisciplinary International Education Research Journal (AMIERJ)

(Bi-Monthly)

Peer-Reviewed Journal

Impact factor:0.948

**Chief-Editor**

**Ubale Amol Baban**

## PEER-REVIEWED INTERNATIONAL JOURNAL

**[Editorial/Head Office: 108, Gokuldham Society, Dr.Ambedkar chowk, Near TV Towar,Badlapur, MSief**

Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)
Peer-Reviewed Journal Vol No III Issues III
JUNE-JULY- 2014 ISSN 2278-5655

# $q$-ANALOGUE OF THE SHIFT OPERATORS AND PSEUDO-POLYNOMIALS OF FRACTIONAL ORDER

**Mohammad Asif[1] and Anju Gupta[2]**
Department of Mathematics
Kalindi College
University of Delhi
New Delhi-110008, India.

**Abstract:** The present paper envisage the $q$-analogues of the operators, results determined by Dattoli et.al, to deal with the families of pseudo-Kampé de Fériet polynomials, which can be viewed as the complement for the theory of $q$-fractional derivatives and $q$-partial fractional differential equations of evolutive type. We show that these families allow the possibility of treating a large variety of $q$-exponential operators providing generalized fractional forms of $q$-shift operators.

**Keywords:** $q$-Calculus, $q$-Hermite-Kampé de Fériet (or Gould-Hopper) polynomials; $q$-Bessel functions; $q$-Exponential operators; Fractional calculus.

**2000 Mathematics Subject Classification:** 33D05, 33C45, 47G20, 26A33.

## 1. Introduction

In this paper, the following notations shall be used (see[9], [10]) for $|\ q\ |< 1$ we have

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) \tag{1.1}$$

$$(a;q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}, \tag{1.2}$$

and for an arbitrary complex number $n$, we have

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1.2a}$$

[1]mohdasiff@gmail.com.
[2]anjjuguptaa@rediffmail.com.

in particular if $n = 1, 2, \cdots,$ we get

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \tag{1.2b}$$

In 1979, Cigler [2], defined the sum of $q$-numbers in the following manner

$$[\alpha]_q \oplus_q [\beta]_q = [\alpha]_q + q^\alpha[\beta]_q = \frac{1-q^\alpha}{1-q} + q^\alpha\frac{1-q^\beta}{1-q} = \frac{1-q^{\alpha+\beta}}{1-q} = [\alpha+\beta]_q \tag{1.3}$$

The $q$-derivative is defined as

$$D_q f(x) = \begin{cases} \dfrac{f(x)-f(qx)}{(1-q)x}, & x \neq 0, \\ f'(0), & x = 0. \end{cases} \tag{1.4}$$

its $n$th iterate is written as

$$D_q^n f(x) = D_q^{n-1}(D_q f(x)),$$

for $n = 1, 2, \cdots,$ where $D_q^0$ denotes the identity operator.
$q$-Gamma function is defined (see[9], [10]) as

$$\Gamma_q(x) = \int_0^\infty u^{x-1} e_q^{-u} d_q u = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1 \tag{1.5}$$

The $q$-Exponential function is defined as follows

$$e_q^x = \sum_{r=0}^\infty \frac{x^r}{[r]_q!}. \tag{1.6}$$

In what follows, we consider analytic function $f(x)$ so that the corresponding $q$-analogue of Taylor expansion

$$f(x+\lambda) = \sum_{k=0}^\infty \frac{\lambda^k}{[k]_q!}\left[\frac{d}{dx}\right]_q^k f(x), \tag{1.7}$$

converges to corresponding value of $f$ in a suitable neighborhood of $x$.

In 2003, Dattoli et al. [3] discussed the exponential operators, In the present paper we find the $q$-analogue of the exponential operators, obtained by Dattoli et. al [3]

$$\widehat{E}_q^m = e_q^{\lambda\left[\frac{\partial}{\partial x}\right]_q^m} \tag{1.8}$$

In particular when $m = 1$, it reduces to the ordinary $q$-shift operator, while for $m = 2$ it can be identified with the $q$-operatorial version of the Gauss transform. Let us write $q$-Taylor transformation, we have

$$e_q^{\lambda\left[\frac{\partial}{\partial x}\right]_q} f(x) = \sum_{r=0}^\infty \frac{\lambda^r\left[\frac{\partial}{\partial x}\right]_q^r}{[r]_q!} f(x) = f(x+\lambda) \tag{1.9}$$

Let us write $q$-analogue of the identity [19], we have

$$e_q^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2+2b\xi} d_q\xi,$$

we find

$$e_q^{\lambda\left[\frac{\partial^2}{\partial x^2}\right]_q} f(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2+2\sqrt{\lambda}\xi\left[\frac{\partial}{\partial x}\right]_q} f(\xi) d_q\xi$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} f(x+2\xi\sqrt{\lambda}) d_q\xi$$

or

$$e_q^{\lambda\left[\frac{\partial^2}{\partial x^2}\right]_q} f(x) = \frac{1}{2\sqrt{\pi\lambda}} \int_{-\infty}^{\infty} e_q^{-\frac{(x-\xi)^2}{4\lambda}} f(\xi) d_q\xi. \qquad (1.10)$$

after a suitable change of variables.

Both the aforementioned eqs. (1.9) and (1.10) are solution of the $q$-partial differential equation:

$$\begin{cases} \left[\frac{\partial}{\partial \lambda}\right]_q F(x,\lambda) = \left[\frac{\partial}{\partial x}\right]_q^m F(x,\lambda), \\ F(x,0) = f(x), \quad m = 1,2. \end{cases} \qquad (A)$$

In case when $m > 2$, the exponential operator $\widehat{E}_q^m = e_q^{\lambda\left[\frac{\partial}{\partial x}\right]_q^m}$ provides formal solution for the generalized heat equation. Therefore, we emphasize that the $q$-analogue of [3] Hermite-Kampé de Fériet polynomials [1] of the type

$$e_q^{\lambda\left[\frac{\partial}{\partial x}\right]_q^m} x^n = \sum_{r=0}^{\infty} \frac{\lambda^r}{[r]_q!} \left[\frac{\partial^{mr}}{\partial x^{mr}}\right]_q x^n$$

$$= [n]_q! \sum_{r=0}^{[\frac{n}{2}]} \frac{x^{n-2r} y^r}{[n-2r]_q! [r]_q!} = H_n^{(m)}(x,y;q) = g_n^m(x,y;q)$$

or equivalently the $q$-analogue GouldHopper polynomials [13, p. 76, eq. (1.9) (6)]:

$$g_n^m(x,y;q) = \sum_{r=0}^{[\frac{n}{2}]} \frac{[n]_q!}{[r]_q! [n-2r]_q!} x^{n-2r} y^r = H_n^{(m)}(x,y;q)$$

are a solution of

$$\begin{cases} \left[\frac{\partial}{\partial \lambda}\right]_q F(x,\lambda) = \left[\frac{\partial}{\partial x}\right]_q^m F(x,\lambda), \\ F(x,0) = x^n. \end{cases} \qquad (A')$$

or in other words[12] and [13]

$$e_q^{\lambda\left[\frac{\partial}{\partial x}\right]_q^m} x^n = H_n^{(m)}(x, y; q) \tag{1.11}$$

This last result is particularly important, since it allows the conclusion that if $f(x)$ is an analytic function defined by the series expansion

$$f(x) = \sum_n c_n x^n$$

then, by $q$-analogue of the Taylor Theorem, we write

$$e_q^{\lambda\left[\frac{\partial}{\partial x}\right]_q^m} f(x) = \sum_n c_n H_n^{(m)}(x, y; q). \tag{1.12}$$

The polynomials $H_n^{(m)}(x, y; q)$ will said to be the $q$-analogue of the Hermite polynomials of index $n$ and order $m$.

**Particular case:** When $q \longmapsto 1$, the equations of this section give raise to the eqs. given in the first section of Dattoli et al. [3].

## 2. Generalized $q$-Exponential Operators

In 2003, extensive uses of exponential operators were used by Dattoli et al. [3]. Let us see the $q$-analogue of the exponential shift operators used in [12] and [13], which play crucial role in the $q$-analogue of the problems concerning in the pure and applied Mathematics [4].

$$\widehat{E}_q = e_q^{\lambda p(x)\left[\frac{d}{dx}\right]_q} \tag{2.1}$$

The properties of the generalized $q$-shift operator are similar to that of discussed in ref. [16-18] and their importance for the solution of generalized $q$-difference equations are similar to that of stressed in ref [5]. The action of $\widehat{E}_q$ on a given function $f(x)$ has been shown to be provided by [12] and [13]

$$\widehat{E}_q f(x) = f[F^{-1}(\lambda + F(x))], \tag{2.2}$$

where

$$F(x) = \int^x \frac{d_q\xi}{p(\xi)} \quad \text{or} \quad \left[\frac{d}{dx}\right]_q F(x) = \frac{1}{p(x)} \quad \text{or} \quad p(x)\left[\frac{d}{dx}\right]_q F(x) = 1 \tag{2.3}$$

defines the associated characteristic function of the generalized $q$-shift operator and $F^{-1}(\cdot)$ is its inverse. The proof of the above identity can be easily given, by Taylor Theorem noting that

$$\widehat{E}_q F(x) = F(x) + \lambda$$

**Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)**
Peer-Reviewed Journal Vol No III Issues III
**JUNE-JULY- 2014 ISSN 2278-5655**

only if [12] and [13]

$$\left[p(x)\left[\frac{d}{dx}\right]_q, F(x)\right] = 1,$$

where $[\cdot,\cdot]$ denotes commutation brackets, that is:

$$p(x)\left[\frac{d}{dx}\right]_q F(x) = 1.$$

More generally, we can always write [8] and [9]

$$\widehat{E_q} f(x) = f[F^{-1}(\lambda + F(x))],$$

It is evident that for $p(x) = 1$, $\widehat{E_q}$ reduces to the ordinary shift operator, when we put $p(x) = x$ we find $F(x) = \ln_q(x)$ and $F^{-1}(x) = e_q^x$, we have

$$e_q^{\lambda x\left[\frac{d}{dx}\right]_q} f(x) = f(e_q^{\lambda} x) \tag{2.4}$$

It is evident that the $q$-analogue of the operators [12] and [13]

$$\widehat{T}_{x;q} = p(x)\left[\frac{d}{dx}\right]_q \tag{2.5}$$

can be viewed as an ordinary $q$-derivative, although $F(x)$ is a function, $[F(x)]^n$ behaves, under the action of $\widehat{T}_{x;q}$, as an ordinary monomial, we obtain indeed

$$\widehat{T}_{x;q}[F(x)]^n = [n]_q[F(x)]^{n-1}, \tag{2.6}$$

we can take advantage from this trivial property to discuss the rule associated with the use of operators like

$$\widehat{E_q}^m = e_q^{\lambda\left(\widehat{T}_{x;q}\right)^m} = e_q^{\lambda\left(p(x)\left[\frac{d}{dx}\right]_q\right)^m}, \tag{2.7}$$

for $m$ integer or real.

According to the conclusion of the introductory section and to these last relations, we can introduce the polynomials

$$h_n^{(m)}(x, y; q) = H_n^{(m)}(F(x), y; q), \tag{2.8}$$

which satisfy the recurrences

$$\left[F(x) \oplus [m]_q y \left[\frac{\partial}{\partial x}\right]_q^{m-1}\right] H_n^{(m)}(F(x), y; q) = [n]_q! \sum_{r=0}^{\left[\frac{n+1}{m}\right]_q} \frac{[n+1-mr]_q[F(x)]^{n+1-mr}}{[n+1-mr]_q!} \frac{y^r}{[r]_q!}$$

**Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)**
Peer-Reviewed Journal Vol No III Issues III
**JUNE-JULY- 2014 ISSN 2278-5655**

$$\oplus[m]_q\left[[n]_q!\sum_{r=1}^{[\frac{n}{m}]}\frac{[F(x)]^{n+1-m(r+1)}}{[n+1-m(r+1)]_q!}\frac{[r+1]_q y^{r+1}}{[r+1]_q!}\right]$$

$$=[n]_q!\sum_{r=0}^{[\frac{n+1}{m}]}\frac{[n+1-mr]_q[F(x)]^{n+1-mr}}{[n+1-mr]_q!}\frac{y^r}{[r]_q!}\oplus[m]_q\left[[n]_q!\sum_{r=0}^{[\frac{n+1}{m}]}\frac{[F(x)]^{n+1-mr}}{[n+1-mr]_q!}\frac{[r]_q y^r}{[r]_q!}\right]$$

$$=[n]_q!\left[\sum_{r=0}^{[\frac{n+1}{m}]}\frac{[F(x)]^{n+1-mr}}{[n+1-mr]_q!}\frac{y^r}{[r]_q!}\right]\{[n+1-mr]_q\oplus[m]_q[r]_q\}$$

The product of $q$-numbers is defined [8; eq.(324)] by $[a]_q = e_q^{\ln_q(a)}$, this power function allows to write

$$[m]_q[r]_q = [mr]_q$$

$$=[n+1]_q[n]_q!\sum_{r=0}^{[\frac{n+1}{m}]}\frac{y^r[F(x)]^{n+1-mr}}{[r]_q![n+1-mr]_q!}$$

or

$$\left[F(x)\oplus[m]_q y\left[\frac{\partial}{\partial x}\right]_q^{m-1}\right]H_n^{(m)}(F(x),y;q) = H_{n+1}^{(m)}(F(x),y;q),$$

by using the eq. (2.8), we have

$$\left.\begin{array}{l}\left[F(x)\oplus[m]_q y\left(\widehat{\mathcal{T}}_{x;q}\right)^{m-1}\right]h_n^{(m)}(x,y;q)=h_{n+1}^{(m)}(x,y;q),\\ \text{and}\\ \widehat{\mathcal{T}}_{x;q}[h_n^{(m)}(x,y;q)]=[n]_q h_{n-1}^{(m)}(x,y;q).\end{array}\right\} \quad (2.9)$$

Clearly $h_n^{(m)}(x,y;q)$ are $q$-functions satisfying $q$-polynomial type identities and will therefore be called pseudo $q$-analogue of H.K.d.F..

It becomes also evident that identities of the following type

$$\left[F(x)+2y\left(\widehat{\mathcal{T}}_{x;q}\right)\right]^n=\sum_{s=0}^{n}\left[\begin{array}{c}n\\ s\end{array}\right]_q H_{n-s}(F(x),y;q)\left(2y(\widehat{\mathcal{T}}_{x;q})\right)^s. \quad (2.10)$$

We show that eq. (2.10) follows from the Weyl identity. Note that since $F(x)$ and $2y(\widehat{\mathcal{T}}_{x;q})$ do not commute, therefore the use of the Newton binomial formula is not allowed. Multiplying the left-hand side of eq. (2.10) by $\frac{t^n}{[n]_q!}$ and summing over $n$, we find

$$\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\left[F(x)+2y\left(\widehat{\mathcal{T}}_{x;q}\right)\right]^n=e_q^{t\left(F(x)+2y\widehat{\mathcal{T}}_{x;q}\right)},$$

by using the Weyl identity, where $\widehat{P} = tF(x)$ and $\widehat{Q} = 2yt\widehat{\mathcal{T}}_{x;q}$

since $$[[\widehat{P},\widehat{Q}],\widehat{P}] = [[\widehat{P},\widehat{Q}],\widehat{Q}] = 0$$

further noting that
$$[\widehat{P},\widehat{Q}] = \widehat{P}\widehat{Q} - \widehat{Q}\widehat{P} = -2yt^2,$$

and
$$e_q^{\widehat{P}+\widehat{Q}} = e_q^{\widehat{P}} e_q^{\widehat{Q}} e_q^{-\frac{1}{2}[\widehat{P},\widehat{Q}]}$$

therefore, we can write
$$e_q^{t(F(x)+2yt\widehat{\mathcal{T}}_{x;q})} = e_q^{tF(x)+yt^2} e_q^{2yt\widehat{\mathcal{T}}_{x;q}}$$

By expanding the $q$-exponential function, we obtain
$$\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}H_n(F(x),y;q)\sum_{s=0}^{\infty}\frac{t^s}{[s]_q!}(2y\widehat{\mathcal{T}}_{x;q})^s = \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\frac{t^{n+s}}{[n]_q![s]_q!}H_n(F(x),y;q)(2y\widehat{\mathcal{T}}_{x;q})^s.$$

Setting $k = n + s$ and inverting summations, we find
$$\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}(F(x)+2y\widehat{\mathcal{T}}_{x;q})^k = \sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}\sum_{s=0}^{k}\begin{bmatrix} k \\ s \end{bmatrix}_q H_{k-s}(F(x),y;q)(2y\widehat{\mathcal{T}}_{x;q})^s.$$

Therefore, (2.10) follows from the comparison of the coefficients of $\frac{t^k}{[k]_q!}$ in the last equation. By using the eq. (2.8) we find the following result
$$\left[F(x)+2y\left(\widehat{\mathcal{T}}_{x;q}\right)\right]^n = \sum_{s=0}^{n}(2y)^s\begin{bmatrix} n \\ s \end{bmatrix}_q h_{n-s}^{(2)}(x,y;q)\left(\widehat{\mathcal{T}}_{x;q}\right)^s \qquad (2.11)$$

and
$$e_q^{y(\widehat{\mathcal{T}}_{x;q})^m} f(F(x)) = e_q^{y(\widehat{\mathcal{T}}_{x;q})^m} f(F(x)) = f\left(F(x)+[m]_q y\left(\widehat{\mathcal{T}}_{x;q}\right)^{m-1}\right)e_q^{y(\widehat{\mathcal{T}}_{x;q})^m} \qquad (2.12)$$

which realize an extension of the ordinary Burchnall and Crofton [5, 16] identities valid for $p(x) = 1$.

It is evident that all the wealth of properties of H.K.d.F. can be extended fairly straightforwardly to the functions $h_n^{(m)}(x,y;q)$. The use of the previously discussed rules may greatly simplify the application of different types of exponential polynomials.

To give some examples, we note e.g. that
$$e_q^{y\left[x\frac{\partial}{\partial x}\right]_q^m}(x^n) = e_q^{y\left[x\frac{\partial}{\partial x}\right]^m} e_q^{n\ln_q(x)} = \sum_{r=0}^{\infty}\frac{n^r}{[r]_q!}H_r^{(m)}(\ln_q(x),y;q), \qquad (2.13)$$

and since

$$\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}H_r^{(m)}(x,y;q) = \sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}\sum_{k=0}^{[\frac{r}{m}]}[r]_q!\frac{x^{r-mk}y^k}{[r-mk]_q![k]_q!} = \sum_{r=0}^{\infty}\sum_{k=0}^{[\frac{r}{m}]}\frac{x^{r-mk}y^k t^r}{[r-mk]_q![k]_q!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(xt)^n}{[n]_q!}\frac{(yt^m)^k}{[k]_q!} = \sum_{n=0}^{\infty}\frac{(xt)^n}{[n]_q!}\sum_{k=0}^{\infty}\frac{(yt^m)^k}{[k]_q!} = e_q^{xt}e_q^{yt^m}$$

or

$$\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}H_r^{(m)}(x,y;q) = e_q^{xt+yt^m} \tag{2.14}$$

or, equivalently,

$$\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}g_r^{(m)}(x,y;q) = e_q^{xt+yt^m}, \tag{2.15}$$

by replacing $t$ and $x$ with $n$ and $\ln_q(x)$ respectively. For the aforementioned Gould-Hopper polynomials [17, p. 86, eq. (1.11) (27)], we find that

$$e_q^{y\left(x\left[\frac{\partial}{\partial x}\right]_q\right)^m}(x^n) = x^n e_q^{yn^m}. \tag{2.16}$$

It is now evident that if $f(x)$ is specified by any analytic function $(f(x)=\sum_n c_n x^n)$, then

$$e_q^{y\left(x\left[\frac{\partial}{\partial x}\right]_q\right)^m} f(x) = \sum_n c_n x^n e_q^{yn^m}, \tag{2.17}$$

provided that the last series is convergent. Further comments on this last result will be presented in the concluding section(4).

A further example of $q$-exponential operator is provided by the case

$$p(x) = (x-b)^2, \qquad F(x) = \int^x \frac{d_q\xi}{(\xi-b)^2} = -\frac{1}{(x-b)},$$

for which we find

$$e_q^{y\left[(x-b)^2\left[\frac{\partial}{\partial x}\right]_q\right]^2}\left[\frac{x-b}{x}\right] = e_q^{y\left[(x-b)^2\left[\frac{\partial}{\partial x}\right]_q\right]^2}\left[\frac{x-b}{x}\right]$$

$$= e_q^{y\left[(x-b)^2\left[\frac{\partial}{\partial x}\right]_q\right]^2}\left[\frac{1}{1+\frac{b}{x-b}}\right]$$

$$= e_q^{y\left[(x-b)^2\left[\frac{\partial}{\partial x}\right]_q\right]^2}\sum_{s=0}^{\infty}\left(-\frac{b}{x-b}\right)^s$$

**Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)**
Peer-Reviewed Journal Vol No III Issues III
**JUNE-JULY- 2014** **ISSN 2278-5655**

$$= \sum_{s=0}^{\infty} (b)^s e^{y \ln(a)\left[(x-b)^2\left[\frac{\partial}{\partial x}\right]_q\right]^2} \left(-\frac{1}{x-b}\right)^s$$

let $-\frac{1}{x-b} = t$, therefore $\left[\frac{\partial t}{\partial x}\right]_q = \frac{1}{(x-b)^2}$ and $\left[\frac{\partial}{\partial x}\right]_q = \left[\frac{\partial}{\partial t}\right]_q \left[\frac{\partial t}{\partial x}\right]_q$ $= \frac{1}{(x-b)^2}\left[\frac{\partial}{\partial t}\right]_q$ or $\left[\frac{\partial}{\partial t}\right]_q = (x-b)^2\left[\frac{\partial}{\partial x}\right]_q$

Now

$$\sum_{s=0}^{\infty} (b)^s e_q^{y\left[\frac{\partial}{\partial t}\right]_q^2} t^s = \sum_{s=0}^{\infty} (b)^s H_s(t, y; q)$$

or

$$e_q^{y\left[(x-b)^2\left[\frac{\partial}{\partial x}\right]_q\right]^2} \left[\frac{x-b}{x}\right] = \sum_{s=0}^{\infty} (b)^s H_s\left(-\frac{1}{x-b}, y; q\right). \tag{2.18}$$

**Particular case:** The results determined by Dattoli et al. [3; in 2 section] can be obtained by taking limiit $q \longmapsto 1$, into the equations of this section.

## 3. $q$-Analogue of the Exponential Operators of Fractional Order

In this section we will discuss fractional shift operators of the type

$$\widehat{E}_q^{\mu} = e_q^{\left(p(x)\left[\frac{d}{dx}\right]_q\right)^{\mu}} \tag{3.1}$$

with $p(x) = 1$ and $\mu$ any real number such that $0 < \mu < 1$.

Before entering into the main body of the discussion, we recall that $q$-analogue of the Riemann-Liouville derivative of fractional order $m$ is defined by (see [15]; see also [4, p. 286, eq. (5.1) (8)])

$$\left[\frac{d}{dx}\right]_q^{\nu} f(x) = \frac{1}{\Gamma_q(m-\nu)}\left[\frac{d^m}{dx^m}\right]_q \int_0^x (x-t)^{m-\nu-1} f(t) d_q t, \tag{3.2}$$

where $m$ is a positive integer such that $m - 1 < \nu < m$.

Accordingly, we get

$$e_q^{\left[\frac{\partial}{\partial x}\right]_q^{\mu}} x^n = H_n^{(\mu)}(x, y; q) \tag{3.3}$$

where $H_n^{(\mu)}(x, y; q)$ are $q$-analogue of the H.K.d.F. pseudo-polynomials of fractional order, defined by

$$H_n^{(\mu)}(x, y; q) = [n]_q! \sum_{r=0}^{\infty} \frac{x^{n-\mu r} y^r}{\Gamma_q(n-\mu r+1)[r]_q!}, \tag{3.4}$$

**Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)**
Peer-Reviewed Journal Vol No III Issues III
**JUNE-JULY- 2014 ISSN 2278-5655**

whose validity easily be checked by the direct expansion of the operator and by the fact that [15]

$$\left(\left[\frac{d}{dx}\right]_q^{\mu}\right)^k x^n = \left[\frac{d}{dx}\right]_q^{k\mu} x^n = \frac{\Gamma_q(n+1)x^{n-\mu r}}{\Gamma_q(n-\mu r+1)[r]_q!}, \quad (\mu < n+1). \tag{3.5}$$

According to the previous discussions $H_n^{(\mu)}(x, y; q)$ is the natural solution of $q$-analogue the fractional Cauchy problem

$$\begin{cases} \left[\frac{\partial}{\partial y}\right]_q u(x,y) = \left[\frac{\partial}{\partial x}\right]_q^{\mu} u(x,y), \\ u(x,0) = x^n. \end{cases} \tag{B}$$

We must underline that the function $H_n^{(\mu)}(x, y; q)$ is an extension of $q$-analogue the ordinary H.K.d.F. or Gould-Hopper polynomials.

More generally we can solve the problem $(B)$ with the general condition

$$u(x,0) = f(x) = \sum_n c_n x^n \tag{3.6}$$

according to the following relation

$$u(x,y) = f(x) = \sum_n c_n H_n^{(\mu)}(x,y;q). \tag{3.7}$$

Clearly, $q$-analogue of the generalized shift exponential operators of the type, is written as

$$\widehat{E}_q^{\mu} = e_q^{\left(p(x)\left[\frac{d}{dx}\right]_q\right)^{\mu}}.$$

With this purpose in view, we consider the problem

$$\begin{cases} \left[\frac{\partial}{\partial y}\right]_q u(x,y) = \left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^{\mu} u(x,y), \\ u(x,0) = (F(x))^n, \end{cases} \tag{C}$$

where

$$F(x) = \int^x \frac{d_q\xi}{p(\xi)}.$$

It is fairly natural to write the solution of $(C)$ as follows:

$$u(x,y) = H_n^{(\mu)}(F(x), y; q) \tag{3.8}$$

This last result completes the purposes of the present paper aimed at providing a general framework for the families of $q$-analogue exponential operators.

**Particular case:** By taking limit $q \longmapsto 1$, the eqs. (3.1) to (3.8) reduce to the results due to Dattoli et al. [3; p. 220-221, section 3].

## 4. Concluding Remarks

In the previous section we have seen that the theory of $q$-exponential operators can be conveniently complemented by the use of functions satisfying recurrences of quasi monomial nature.

In these concluding remarks we will discuss the introduction of a family of functions which can be viewed as a fairly natural consequence of the so far developed formalism.

We consider the case of logarithmic Bessel functions, whose generating function can be cast in the following form

$$G(x,\vartheta;q) = x^{i\sin_q(\vartheta)} = \sum_{n=-\infty}^{\infty} e_q^{in\vartheta} J_n(\ln_q(x);q), \tag{4.1}$$

where $J_n(x)$ denote the first kind cylinder Bessel functions, that is its $q$-analogue is written as

$$J_n(z;q) \;=\; \frac{(q^{n+1};q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}z\right)^n \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^k}{(q;q)_k(q^{n+1};q)_k}.$$

It is evident that we can take advantage from the discussion of the previous sections, to consider the following problem

$$e^{y\left(x\left[\frac{\partial}{\partial x}\right]_q\right)^2} x^{i\sin_q(\vartheta)} \;=\; e_q^{i\sin_q(\vartheta)\left(\ln_q(x)+2yx\left[\frac{\partial}{\partial x}\right]_q\right)}. \tag{4.2}$$

The $q$-exponential can be decoupled by means of the Weyl rule

$$e_q^{\widehat{P}+\widehat{Q}} = e_q^{\widehat{P}} e_q^{\widehat{Q}} e_q^{-\frac{1}{2}[\widehat{P},\widehat{Q}]},$$

where

$$[\widehat{P},\widehat{Q}] = \widehat{P}\widehat{Q} - \widehat{Q}\widehat{P},$$

by setting indeed

$$\left.\begin{aligned} \widehat{P} &= i\sin_q(\vartheta)\ln_q(x), \\ \widehat{Q} &= 2iy(\sin_q(\vartheta))x\left[\tfrac{\partial}{\partial x}\right]_q, \end{aligned}\right\} \tag{4.3}$$

we find

$$[\widehat{P},\widehat{Q}] = 2y[\sin_q(\vartheta)]^2, \tag{4.4}$$

**Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)**
Peer-Reviewed Journal Vol No III Issues III
**JUNE-JULY- 2014** **ISSN 2278-5655**

thus getting

$$e_q^{y\left(x\left[\frac{\partial}{\partial x}\right]_q\right)^2} x^{i\sin_q(\vartheta)} = x^{i\sin_q(\vartheta)} e_q^{-y[\sin_q(\vartheta)]^2}. \tag{4.5}$$

Which is the generating function of a two-variable Bessel function, namely

$$x^{i\sin_q(\vartheta)} e_q^{-y[\sin_q(\vartheta)]^2} = \sum_{n=-\infty}^{\infty} e_q^{in\vartheta} ({}_hJ_n(x,y;q)). \tag{4.6}$$

$${}_hJ_n(x,y;q) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(\ln_q(x),y;q)}{2^{n+2r}[r]_q![n+r]_q!}. \tag{4.7}$$

It is evident that we ended up with a Bessel type function generalizing those of Hermite nature discussed in ref. [8]. It is worth emphasizing that the above equations satisfy a partial differential equation of the type

$$\begin{cases} \left[\frac{\partial}{\partial y}\right]_q ({}_hJ_n(x,y;q)) = \left(x\left[\frac{\partial}{\partial x}\right]_q\right)^2 ({}_hJ_n(x,y;q)), \\ {}_hJ_n(x,0;q) = J_n(\ln_q(x);q). \end{cases} \tag{4.8}$$

It is evident that the above considerations can be extended to any generating function of the type [3]

$$e^{iF(x)\sin_q(\vartheta)}. \tag{4.9}$$

Before concluding we will show how the combined use of integral transform and the previous formalism allows the derivation of further important relations. According to the identity [19]

$$e_q^{\lambda\hat{P}^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2+2\xi\sqrt{\lambda}\hat{P}} d_q\xi. \tag{4.10}$$

we can easily conclude that the polynomials $h_n^{(2)}(x,y;q)$ can also be realized in terms of the integral representation

$$e_q^{y\left(p(x)\left[\frac{\partial}{\partial x}\right]_q\right)^2} [F(x)]^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} [F(x)+2\sqrt{y}\xi]^n d_q\xi. \tag{4.11}$$

Going back to eq. (2.17) and specializing for $m = 2$, the use of the above relations allows to state the following identity

$$\sum_{n=0}^{\infty} \frac{(-1)^n e_q^{-yn^2}}{[n]_q!} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} e_q^{-\cos_q(2\sqrt{y}\xi)} \cos_q(\sin_q(2\sqrt{y}\xi)) d_q\xi.$$

Finally since [7]

$$e_q^{-yd} = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} e_q^{-d^2 t} d_q t \tag{4.12}$$


www.aarhat.com Impact Factor 0.948

Aarhat Multidisciplinary International Education Research Journal (AMIERJ) (Bi-Monthly)
Peer-Reviewed Journal Vol No III Issues III
JUNE-JULY- 2014 ISSN 2278-5655

by replacing $d$ with $\mathcal{T}_{x;q}^{\frac{1}{2}}$ and by using the previously discussed rules we find

$$e_q^{-y(\mathcal{T}_{x;q})^{\frac{1}{2}}} = \frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} e_q^{-(\mathcal{T}_{x;q})t} d_q t$$

let us see the effects of $e_q^{-y(\mathcal{T}_{x;q})^{\frac{1}{2}}}$ on an ordinary function, we have

$$e_q^{-y(\mathcal{T}_{x;q})^{\frac{1}{2}}} f(x) = \frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} e_q^{-t(\mathcal{T}_{x;q})} f(x) d_q t$$

or

$$e_q^{-y(\mathcal{T}_{x;q})^{\frac{1}{2}}} f(x) = \frac{y}{2\sqrt{\pi}}\int_0^\infty \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} f(F^{-1}(F(x)-t)) d_q t. \tag{4.13}$$

Which realizes the transform providing the action of a fractional generalized shift operator on a given function, the validity of both (3.11) and (3.12) is limited to the case in which the integral converges.

**Particular case:** Taking the limit $q \longmapsto 1$, the eqs. (4.1) to (4.13) give raise to the eqs. (15) to (26) due to Dattoli et al. [2].

## References


[1]. Appell, P.; Kampé de Fériet, J.; *Fonctions hypergéométriques et hypersphériques. Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2]. Asif, M.; *q-Analogue of Exponential Operators and Difference Equations*, Kyungpook Math. J. **53**(2013), pp. 349-369

[3]. Asif, M.; Gupta, Anju; *Generalized Exponential Operators and Difference Equations*, Italian journal of Pure and Applied Mathematics no. 30 (to appear).

[4]. Asif, M.; Gupta, Anju; *q-Analogue of the Operational Methods, Fractional Operators and Special Polynomials*, communicated for the publication.

[5]. Cigler J.; *Operatormethoden für q-Identitaten. Monatshefte für Mathematik* 88, (1979), 87-105.

[6]. Dattoli, G.; Ricci, P.E.; Sacchetti, D.; *Generalized shift operators and pseudo-polynomials of fractional order* Applied Mathematics and Computation **141** (2003) 215-224.

[7]. Dattoli, G.; Gallardo, J.C.; Torre, A.; *An algebraic point of view to operatorial methods in optics*, Riv. Nuovo Cimento **11** (1988) 1-85

[8]. Dattoli, G.; Ottaviani, P. L.; Torre, A.; Vázquez, L.; *Evolution operator equations: integration with algebraic and finite difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory,* Riv. Nuovo Cimento, **2** (1997), 1-133.

[9]. Dattoli, G.; Levi, D.; *Exponential operators and generalized difference equations,* Riv. Nuovo Cimento B, **115** (2000), 653-662.

[10]. Dattoli, G.; Ricci, P.E.; Cesarano, C.; Vázquez, L.; *Fractional derivatives: integral representations and generalized polynomials,* J. Concrete Appl. Math.**2** (2004), no. 1, 59-66.

[11]. Dattoli, G.; *HermiteBessel and LaguerreBessel functions: A by-product of the monomiality principle, in: D. Cocolicchio, G. Dattoli, H.M. Srivastava (Eds.),* Advanced Special Functions and Applications, Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics, Melfi, 912 May 1999, Aracne Editrice, Rome, 2000, pp. 147-164.

[12]. Ernest, T.; *The History of q- Calculus and a New Method,* Licentiate Thesis, Department of Mathematics, Uppsala University, SE-75106 Uppsala, Sweden.

[13]. Gasper, G.; Rahman, M.; *Basic Hypergeometric Series,* Cambridge University Press, Cambridge, 1990, 2004.

[14]. Kac, V.; Cheung, P.; *Quantum Calculus,* New York, 2002.

[15]. Khan, M. A.; Asif, M.; *Shift Operators on the Base* $a(a > 0, \neq 1)$ *and Monomial Type Functions,* International Transactions in Mathematical Sciences and Computer, July-December 2009, Volume 2, No. 2, pp. 453-462.

[16]. Miller, K.S.; Ross, B.; *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York, 1993.

[17]. Ricci, P.E.; Tecniche operatoriali e Funzioni speciali, Preprint 2001.

[18]. Srivastava, H.M.; Manocha, H.L.; *A Treatise on Generating Functions,* Wiley, New York, 1984.

[19]. Wilcox, R.M.; *Exponential operators and parameter differentiation in quantum physics,* J. Math. Phys. **8** (1967) 962-982.

[20]. Witschel, W.; *On Baker-Campbell-Hausdorff operator disentangling by similarity transformations,* Phys. Lett. A, **111** (1985), 383-388.

[21]. Wright, E.M.; *The generalized Bessel function of order greater than one,* Quart. J. Math. Oxford Ser. **11** (1940) 36-48.