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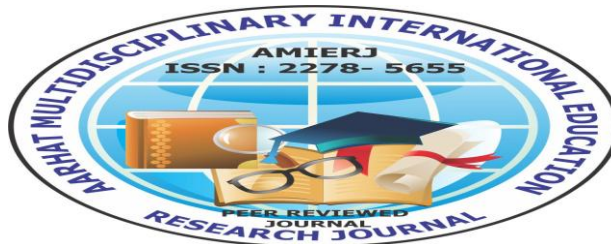
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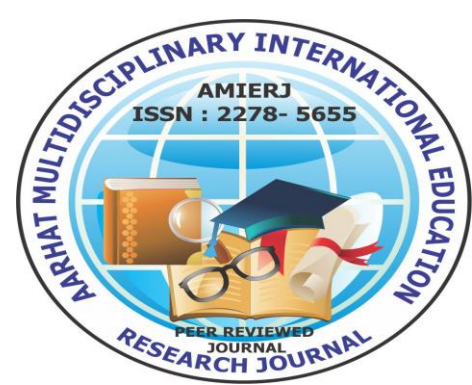
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q-ANALOGUE OF THE SHIFT OPERATORS AND PSEUDO-POLYNOMIALS OF FRACTIONAL ORDER

Mohammad Asif¹ and Anju Gupta²
 Department of Mathematics
 Kalindi College
 University of Delhi
 New Delhi-110008, India.

Abstract: The present paper envisage the q -analogues of the operators, results determined by Dattoli et.al, to deal with the families of pseudo-Kampé de Fériet polynomials, which can be viewed as the complement for the theory of q -fractional derivatives and q -partial fractional differential equations of evolutive type. We show that these families allow the possibility of treating a large variety of q -exponential operators providing generalized fractional forms of q -shift operators.

Keywords: q -Calculus, q -Hermite-Kampé de Fériet (or Gould-Hopper) polynomials; q -Bessel functions; q -Exponential operators; Fractional calculus.

2000 Mathematics Subject Classification: 33D05, 33C45, 47G20, 26A33.

1. Introduction

In this paper, the following notations shall be used (see[9], [10]) for $|q| < 1$ we have

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j) \tag{1.1}$$

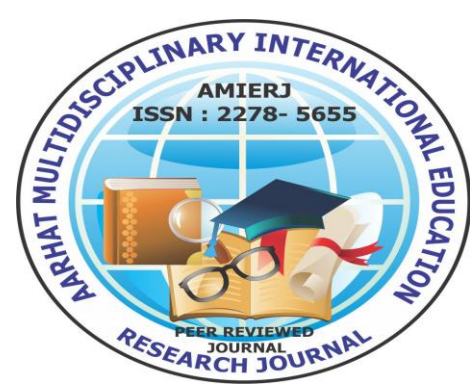
$$(a; q)_n = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}, \tag{1.2}$$

and for an arbitrary complex number n , we have

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \tag{1.2a}$$

¹mohdasiff@gmail.com.

²anjjuguptaa@rediffmail.com.



in particular if $n = 1, 2, \dots$, we get

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \quad (1.2b)$$

In 1979, Cigler [2], defined the sum of q -numbers in the following manner

$$[\alpha]_q \oplus_q [\beta]_q = [\alpha]_q + q^\alpha [\beta]_q = \frac{1 - q^\alpha}{1 - q} + q^\alpha \frac{1 - q^\beta}{1 - q} = \frac{1 - q^{\alpha+\beta}}{1 - q} = [\alpha + \beta]_q \quad (1.3)$$

The q -derivative is defined as

$$D_q f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1 - q)x}, & x \neq 0, \\ f'(0), & x = 0. \end{cases} \quad (1.4)$$

its n th iterate is written as

$$D_q^n f(x) = D_q^{n-1}(D_q f(x)),$$

for $n = 1, 2, \dots$, where D_q^0 denotes the identity operator. q -Gamma function is defined (see[9], [10]) as

$$\Gamma_q(x) = \int_0^\infty u^{x-1} \varepsilon_q^{-u} d_q u = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1 \quad (1.5)$$

The q -Exponential function is defined as follows

$$e_q^x = \sum_{r=0}^\infty \frac{x^r}{[r]_q!}. \quad (1.6)$$

In what follows, we consider analytic function $f(x)$ so that the corresponding q -analogue of Taylor expansion

$$f(x + \lambda) = \sum_{k=0}^\infty \frac{\lambda^k}{[k]_q!} \left[\frac{d}{dx} \right]_q^k f(x), \quad (1.7)$$

converges to corresponding value of f in a suitable neighborhood of x .

In 2003, Dattoli et al. [3] discussed the exponential operators, In the present paper we find the q -analogue of the exponential operators, obtained by Dattoli et. al [3]

$$\widehat{E}_q^m = e_q^{\lambda \left[\frac{\partial}{\partial x} \right]_q^m} \quad (1.8)$$

In particular when $m = 1$, it reduces to the ordinary q -shift operator, while for $m = 2$ it can be identified with the q -operatorial version of the Gauss transform. Let us write q -Taylor transformation, we have

$$e_q^{\lambda \left[\frac{\partial}{\partial x} \right]_q} f(x) = \sum_{r=0}^\infty \frac{\lambda^r \left[\frac{\partial}{\partial x} \right]_q^r f(x)}{[r]_q!} = f(x + \lambda) \quad (1.9)$$

Let us write q -analogue of the identity [19], we have

$$e_q^{b^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2 + 2b\xi} d_q \xi,$$

we find

$$\begin{aligned} e_q^{\lambda \left[\frac{\partial^2}{\partial x^2} \right]_q} f(x) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2 + 2\sqrt{\lambda}\xi \left[\frac{\partial}{\partial x} \right]_q} f(\xi) d_q \xi \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} f(x + 2\xi\sqrt{\lambda}) d_q \xi \end{aligned}$$

or

$$e_q^{\lambda \left[\frac{\partial^2}{\partial x^2} \right]_q} f(x) = \frac{1}{2\sqrt{\pi\lambda}} \int_{-\infty}^{\infty} e_q^{-\frac{(x-\xi)^2}{4\lambda}} f(\xi) d_q \xi. \quad (1.10)$$

after a suitable change of variables.

Both the aforementioned eqs. (1.9) and (1.10) are solution of the q -partial differential equation:

$$\begin{cases} \left[\frac{\partial}{\partial \lambda} \right]_q F(x, \lambda) = \left[\frac{\partial}{\partial x} \right]_q^m F(x, \lambda), \\ F(x, 0) = f(x), \quad m = 1, 2. \end{cases} \quad (A)$$

In case when $m > 2$, the exponential operator $\hat{E}_q^m = e_q^{\lambda \left[\frac{\partial}{\partial x} \right]_q^m}$ provides formal solution for the generalized heat equation. Therefore, we emphasize that the q -analogue of [3] Hermite-Kampé de Fériet polynomials [1] of the type

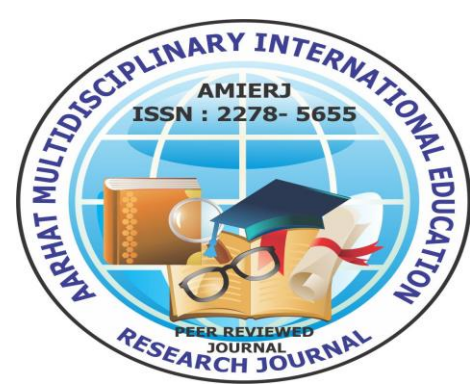
$$\begin{aligned} e_q^{\lambda \left[\frac{\partial}{\partial x} \right]_q^m} x^n &= \sum_{r=0}^{\infty} \frac{\lambda^r}{[r]_q!} \left[\frac{\partial^{mr}}{\partial x^{mr}} \right]_q x^n \\ &= [n]_q! \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{x^{n-2r} y^r}{[n-2r]_q! [r]_q!} = H_n^{(m)}(x, y; q) = g_n^m(x, y; q) \end{aligned}$$

or equivalently the q -analogue GouldHopper polynomials [13, p. 76, eq. (1.9) (6)]:

$$g_n^m(x, y; q) = \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{[n]_q!}{[r]_q! [n-2r]_q!} x^{n-2r} y^r = H_n^{(m)}(x, y; q)$$

are a solution of

$$\begin{cases} \left[\frac{\partial}{\partial \lambda} \right]_q F(x, \lambda) = \left[\frac{\partial}{\partial x} \right]_q^m F(x, \lambda), \\ F(x, 0) = x^n. \end{cases} \quad (A')$$



or in other words [12] and [13]

$$e_q^{\lambda \left[\frac{d}{dx} \right]_q^m} x^n = H_n^{(m)}(x, y; q) \quad (1.11)$$

This last result is particularly important, since it allows the conclusion that if $f(x)$ is an analytic function defined by the series expansion

$$f(x) = \sum_n c_n x^n$$

then, by q -analogue of the Taylor Theorem, we write

$$e_q^{\lambda \left[\frac{d}{dx} \right]_q^m} f(x) = \sum_n c_n H_n^{(m)}(x, y; q). \quad (1.12)$$

The polynomials $H_n^{(m)}(x, y; q)$ will said to be the q -analogue of the Hermite polynomials of index n and order m .

Particular case: When $q \mapsto 1$, the equations of this section give raise to the eqs. given in the first section of Dattoli et al. [3].

2. Generalized q -Exponential Operators

In 2003, extensive uses of exponential operators were used by Dattoli et al. [3]. Let us see the q -analogue of the exponential shift operators used in [12] and [13], which play crucial role in the q -analogue of the problems concerning in the pure and applied Mathematics [4].

$$\widehat{E}_q = e_q^{\lambda p(x) \left[\frac{d}{dx} \right]_q} \quad (2.1)$$

The properties of the generalized q -shift operator are similar to that of discussed in ref. [16-18] and their importance for the solution of generalized q -difference equations are similar to that of stressed in ref [5]. The action of \widehat{E}_q on a given function $f(x)$ has been shown to be provided by [12] and [13]

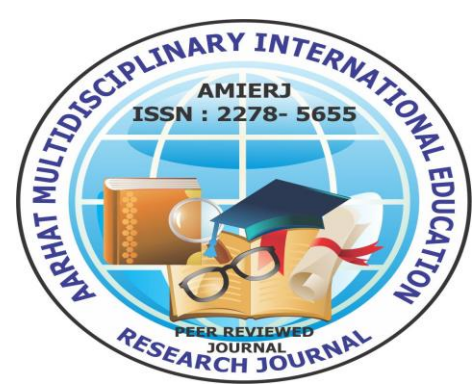
$$\widehat{E}_q f(x) = f[F^{-1}(\lambda + F(x))], \quad (2.2)$$

where

$$F(x) = \int^x \frac{d_q \xi}{p(\xi)} \text{ or } \left[\frac{d}{dx} \right]_q F(x) = \frac{1}{p(x)} \text{ or } p(x) \left[\frac{d}{dx} \right]_q F(x) = 1 \quad (2.3)$$

defines the associated characteristic function of the generalized q -shift operator and $F^{-1}(\cdot)$ is its inverse. The proof of the above identity can be easily given, by Taylor Theorem noting that

$$\widehat{E}_q F(x) = F(x) + \lambda$$



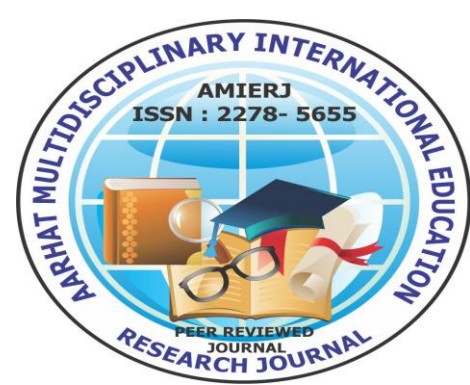
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only if [12] and [13]

$$\left[p(x) \left[\frac{d}{dx} \right]_q, F(x) \right] = 1,$$

where $[\cdot, \cdot]$ denotes commutation brackets, that is:

$$p(x) \left[\frac{d}{dx} \right]_q F(x) = 1.$$

More generally, we can always write [8] and [9]

$$\widehat{E}_q f(x) = f[F^{-1}(\lambda + F(x))],$$

It is evident that for $p(x) = 1$, \widehat{E}_q reduces to the ordinary shift operator, when we put $p(x) = x$ we find $F(x) = \ln_q(x)$ and $F^{-1}(x) = e_q^\lambda$, we have

$$e_q^{\lambda x \left[\frac{d}{dx} \right]_q} f(x) = f(e_q^\lambda x) \quad (2.4)$$

It is evident that the q -analogue of the operators [12] and [13]

$$\widehat{T}_{x,q} = p(x) \left[\frac{d}{dx} \right]_q \quad (2.5)$$

can be viewed as an ordinary q -derivative, although $F(x)$ is a function, $[F(x)]^n$ behaves, under the action of $\widehat{T}_{x,q}$, as an ordinary monomial, we obtain indeed

$$\widehat{T}_{x,q}[F(x)]^n = [n]_q[F(x)]^{n-1}, \quad (2.6)$$

we can take advantage from this trivial property to discuss the rule associated with the use of operators like

$$\widehat{E}_q^m = e_q^{\lambda(\widehat{T}_{x,q})^m} = e_q^{\lambda \left(p(x) \left[\frac{d}{dx} \right]_q \right)^m}, \quad (2.7)$$

for m . integer or real.

According to the conclusion of the introductory section and to these last relations, we can introduce the polynomials.

$$h_n^{(m)}(x, y; q) = H_n^{(m)}(F(x), y; q), \quad (2.8)$$

which satisfy the recurrences

$$\left[F(x) \oplus [m]_q y \left[\frac{\partial}{\partial x} \right]_q^{m-1} \right] H_n^{(m)}(F(x), y; q) = [n]_q! \sum_{r=0}^{\left[\frac{n+1}{m} \right]_q} \frac{[n+1 - mr]_q [F(x)]^{n+1-mr}}{[n+1 - mr]_q!} \frac{y^r}{[r]_q!}$$

$$\begin{aligned} & \oplus [m]_q \left[[n]_q! \sum_{r=1}^{\lfloor \frac{n+1}{m} \rfloor} \frac{[F(x)]^{n+1-m(r+1)}}{[n+1-m(r+1)]_q!} \frac{[r+1]_q y^{r+1}}{[r+1]_q!} \right] \\ &= [n]_q! \sum_{r=0}^{\lfloor \frac{n+1}{m} \rfloor} \frac{[n+1-mr]_q [F(x)]^{n+1-mr} y^r}{[n+1-mr]_q! [r]_q!} \oplus [m]_q \left[[n]_q! \sum_{r=0}^{\lfloor \frac{n+1}{m} \rfloor} \frac{[F(x)]^{n+1-mr} [r]_q y^r}{[n+1-mr]_q! [r]_q!} \right] \\ &= [n]_q! \left[\sum_{r=0}^{\lfloor \frac{n+1}{m} \rfloor} \frac{[F(x)]^{n+1-mr} y^r}{[n+1-mr]_q! [r]_q!} \right] \{ [n+1-mr]_q \oplus [m]_q [r]_q \} \end{aligned}$$

The product of q -numbers is defined [8; eq.(324)] by $[a]_q = e_q^{\ln_q(a)}$, this power function allows to write

$$[m]_q [r]_q = [mr]_q$$

$$= [n+1]_q [n]_q! \sum_{r=0}^{\lfloor \frac{n+1}{m} \rfloor} \frac{y^r [F(x)]^{n+1-mr}}{[r]_q! [n+1-mr]_q!}$$

or

$$\left[F(x) \oplus [m]_q y \left[\frac{\partial}{\partial x} \right]_q^{m-1} \right] H_n^{(m)}(F(x), y; q) = H_{n+1}^{(m)}(F(x), y; q),$$

by using the eq. (2.8), we have

$$\left. \begin{aligned} & \left[F(x) \oplus [m]_q y \left(\widehat{\mathcal{T}}_{x;q} \right)^{m-1} \right] h_n^{(m)}(x, y; q) = h_{n+1}^{(m)}(x, y; q), \\ & \text{and} \\ & \widehat{\mathcal{T}}_{x;q} [h_n^{(m)}(x, y; q)] = [n]_q h_{n-1}^{(m)}(x, y; q). \end{aligned} \right\} \quad (2.9)$$

Clearly $h_n^{(m)}(x, y; q)$ are q -functions satisfying q -polynomial type identities and will therefore be called pseudo q -analogue of H.K.d.F..

It becomes also evident that identities of the following type

$$\left[F(x) + 2y \left(\widehat{\mathcal{T}}_{x;q} \right) \right]^n = \sum_{s=0}^n \begin{bmatrix} n \\ s \end{bmatrix}_q H_{n-s}(F(x), y; q) \left(2y \left(\widehat{\mathcal{T}}_{x;q} \right) \right)^s. \quad (2.10)$$

We show that eq. (2.10) follows from the Weyl identity. Note that since $F(x)$ and $2y \left(\widehat{\mathcal{T}}_{x;q} \right)$ do not commute, therefore the use of the Newton binomial formula is not allowed. Multiplying the left-hand side of eq. (2.10) by $\frac{t^n}{[n]_q!}$ and summing over n , we find

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \left[F(x) + 2y \left(\widehat{\mathcal{T}}_{x;q} \right) \right]^n = e_q^{t(F(x)+2y\widehat{\mathcal{T}}_{x;q})},$$

by using the Weyl identity, where $\hat{P} = tF(x)$ and $\hat{Q} = 2yt\hat{T}_{x;q}$

since
$$[[\hat{P}, \hat{Q}], \hat{P}] = [[\hat{P}, \hat{Q}], \hat{Q}] = 0$$

further noting that

$$[\hat{P}, \hat{Q}] = \hat{P}\hat{Q} - \hat{Q}\hat{P} = -2yt^2,$$

and

$$e_q^{\hat{P}+\hat{Q}} = e_q^{\hat{P}} e_q^{\hat{Q}} e_q^{-\frac{1}{2}[\hat{P}, \hat{Q}]}$$

therefore, we can write

$$e_q^{t(F(x)+2yt\hat{T}_{x;q})} = e_q^{tF(x)+yt^2} e_q^{2yt\hat{T}_{x;q}}$$

By expanding the q -exponential function, we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} H_n(F(x), y; q) \sum_{s=0}^{\infty} \frac{t^s}{[s]_q!} (2y\hat{T}_{x;q})^s = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{t^{n+s}}{[n]_q! [s]_q!} H_n(F(x), y; q) (2y\hat{T}_{x;q})^s.$$

Setting $k = n + s$ and inverting summations, we find

$$\sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} (F(x) + 2y\hat{T}_{x;q})^k = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} \sum_{s=0}^k \begin{bmatrix} k \\ s \end{bmatrix}_q H_{k-s}(F(x), y; q) (2y\hat{T}_{x;q})^s.$$

Therefore, (2.10) follows from the comparison of the coefficients of $\frac{t^k}{[k]_q!}$ in the last equation. By using the eq. (2.8) we find the following result

$$[F(x) + 2y(\hat{T}_{x;q})]^n = \sum_{s=0}^n (2y)^s \begin{bmatrix} n \\ s \end{bmatrix}_q h_{n-s}^{(2)}(x, y; q) (\hat{T}_{x;q})^s \tag{2.11}$$

and

$$e_q^{y(\hat{T}_{x;q})^m} f(F(x)) = e_q^{y(\hat{T}_{x;q})^m} f(F(x)) = f(F(x) + [m]_q y (\hat{T}_{x;q})^{m-1}) e_q^{y(\hat{T}_{x;q})^m} \tag{2.12}$$

which realize an extension of the ordinary Burchnell and Crofton [5, 16] identities valid for $p(x) = 1$.

It is evident that all the wealth of properties of H.K.d.F. can be extended fairly straightforwardly to the functions $h_n^{(m)}(x, y; q)$. The use of the previously discussed rules may greatly simplify the application of different types of exponential polynomials.

To give some examples, we note e.g. that

$$e_q^{y \left[\frac{x}{2x} \right]_q^m} (x^n) = e_q^{y \left[\frac{x}{2x} \right]_q^m} e_q^{n \ln_q(x)} = \sum_{r=0}^{\infty} \frac{n^r}{[r]_q!} H_r^{(m)}(\ln_q(x), y; q), \tag{2.13}$$

and since

$$\begin{aligned}
 \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} H_r^{(m)}(x, y; q) &= \sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} \sum_{k=0}^{\lfloor \frac{r}{m} \rfloor} [r]_q! \frac{x^{r-mk} y^k}{[r-mk]_q! [k]_q!} = \sum_{r=0}^{\infty} \sum_{k=0}^{\lfloor \frac{r}{m} \rfloor} \frac{x^{r-mk} y^k t^r}{[r-mk]_q! [k]_q!} \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(xt)^n (yt^m)^k}{[n]_q! [k]_q!} = \sum_{n=0}^{\infty} \frac{(xt)^n}{[n]_q!} \sum_{k=0}^{\infty} \frac{(yt^m)^k}{[k]_q!} = e_q^{xt} e_q^{yt^m}
 \end{aligned}$$

or

$$\sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} H_r^{(m)}(x, y; q) = e_q^{xt+yt^m} \tag{2.14}$$

or, equivalently,

$$\sum_{r=0}^{\infty} \frac{t^r}{[r]_q!} g_r^{(m)}(x, y; q) = e_q^{xt+yt^m}, \tag{2.15}$$

by replacing t and x with n and $\ln_q(x)$ respectively. For the aforementioned Gould-Hopper polynomials [17, p. 86, eq. (1.11) (27)], we find that

$$e_q^{y \left(x \left[\frac{q}{q-x} \right]_q \right)^m} (x^n) = x^n e_q^{yn^m}. \tag{2.16}$$

It is now evident that if $f(x)$ is specified by any analytic function ($f(x) = \sum_n c_n x^n$), then

$$e_q^{y \left(x \left[\frac{q}{q-x} \right]_q \right)^m} f(x) = \sum_n c_n x^n e_q^{yn^m}, \tag{2.17}$$

provided that the last series is convergent. Further comments on this last result will be presented in the concluding section(4).

A further example of q -exponential operator is provided by the case

$$p(x) = (x-b)^2, \quad F(x) = \int^x \frac{d_q \xi}{(\xi-b)^2} = -\frac{1}{(x-b)},$$

for which we find

$$\begin{aligned}
 e_q^{y \left[(x-b)^2 \left[\frac{q}{q-x} \right]_q \right]^2} \left[\frac{x-b}{x} \right] &= e_q^{y \left[(x-b)^2 \left[\frac{q}{q-x} \right]_q \right]^2} \left[\frac{x-b}{x} \right] \\
 &= e_q^{y \left[(x-b)^2 \left[\frac{q}{q-x} \right]_q \right]^2} \left[\frac{1}{1 + \frac{b}{x-b}} \right] \\
 &= e_q^{y \left[(x-b)^2 \left[\frac{q}{q-x} \right]_q \right]^2} \sum_{s=0}^{\infty} \left(-\frac{b}{x-b} \right)^s
 \end{aligned}$$

$$= \sum_{s=0}^{\infty} (b)^s e^{y \ln(a) \left[(x-b)^2 \left[\frac{\partial}{\partial x} \right]_q \right]^2} \left(-\frac{1}{x-b} \right)^s$$

let $-\frac{1}{x-b} = t$, therefore $\left[\frac{\partial t}{\partial x} \right]_q = \frac{1}{(x-b)^2}$ and $\left[\frac{\partial}{\partial x} \right]_q = \left[\frac{\partial}{\partial t} \right]_q \left[\frac{\partial t}{\partial x} \right]_q$

$$= \frac{1}{(x-b)^2} \left[\frac{\partial}{\partial t} \right]_q \text{ or } \left[\frac{\partial}{\partial t} \right]_q = (x-b)^2 \left[\frac{\partial}{\partial x} \right]_q$$

Now

$$\sum_{s=0}^{\infty} (b)^s e_q^{y \left[\frac{\partial}{\partial t} \right]_q^2 t^s} = \sum_{s=0}^{\infty} (b)^s H_s(t, y; q)$$

or

$$e_q^{y \left[(x-b)^2 \left[\frac{\partial}{\partial x} \right]_q \right]^2} \left[\frac{x-b}{x} \right] = \sum_{s=0}^{\infty} (b)^s H_s \left(-\frac{1}{x-b}, y; q \right). \tag{2.18}$$

Particular case: The results determined by Dattoli et al. [3; in 2 section] can be obtained by taking limit $q \rightarrow 1$, into the equations of this section.

3. q -Analogue of the Exponential Operators of Fractional Order

In this section we will discuss fractional shift operators of the type

$$\widehat{E}_q^\mu = e_q^{\left(p(x) \left[\frac{\partial}{\partial x} \right]_q \right)^\mu} \tag{3.1}$$

with $p(x) = 1$ and μ any real number such that $0 < \mu < 1$.

Before entering into the main body of the discussion, we recall that q -analogue of the Riemann-Liouville derivative of fractional order m is defined by (see [15]; see also [4, p. 286, eq. (5.1) (8)])

$$\left[\frac{d}{dx} \right]_q^\nu f(x) = \frac{1}{\Gamma_q(m-\nu)} \left[\frac{d^m}{dx^m} \right]_q \int_0^x (x-t)^{m-\nu-1} f(t) d_q t, \tag{3.2}$$

where m is a positive integer such that $m-1 < \nu < m$.

Accordingly, we get

$$e_q^{\left[\frac{\partial}{\partial x} \right]_q^\mu} x^n = H_n^{(\mu)}(x, y; q) \tag{3.3}$$

where $H_n^{(\mu)}(x, y; q)$ are q -analogue of the H.K.d.F. pseudo-polynomials of fractional order, defined by

$$H_n^{(\mu)}(x, y; q) = [n]_q! \sum_{r=0}^{\infty} \frac{x^{n-\mu r} y^r}{\Gamma_q(n-\mu r+1) [r]_q!}, \tag{3.4}$$

whose validity easily be checked by the direct expansion of the operator and by the fact that [15]

$$\left(\left[\frac{d}{dx} \right]_q^\mu \right)^k x^n = \left[\frac{d}{dx} \right]_q^{k\mu} x^n = \frac{\Gamma_q(n+1)x^{n-\mu k}}{\Gamma_q(n-\mu k+1)[r]_q!}, \quad (\mu < n+1). \quad (3.5)$$

According to the previous discussions $H_n^{(\mu)}(x, y; q)$ is the natural solution of q -analogue the fractional Cauchy problem

$$\begin{cases} \left[\frac{\partial}{\partial y} \right]_q u(x, y) = \left[\frac{\partial}{\partial x} \right]_q^\mu u(x, y), \\ u(x, 0) = x^n. \end{cases} \quad (B)$$

We must underline that the function $H_n^{(\mu)}(x, y; q)$ is an extension of q -analogue the ordinary H.K.d.F. or Gould-Hopper polynomials.

More generally we can solve the problem (B) with the general condition

$$u(x, 0) = f(x) = \sum_n c_n x^n \quad (3.6)$$

according to the following relation

$$u(x, y) = f(x) = \sum_n c_n H_n^{(\mu)}(x, y; q). \quad (3.7)$$

Clearly, q -analogue of the generalized shift exponential operators of the type, is written as

$$\widehat{E}_q^\mu = e_q^{\left(p(x) \left[\frac{d}{dx} \right]_q \right)^\mu}.$$

With this purpose in view, we consider the problem

$$\begin{cases} \left[\frac{\partial}{\partial y} \right]_q u(x, y) = \left(p(x) \left[\frac{\partial}{\partial x} \right]_q \right)^\mu u(x, y), \\ u(x, 0) = (F(x))^n, \end{cases} \quad (C)$$

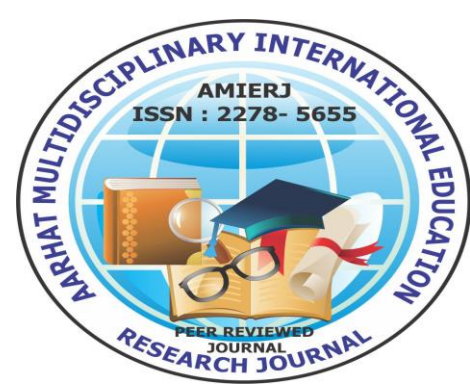
where

$$F(x) = \int_0^\infty \frac{d_q \xi}{p(\xi)}.$$

It is fairly natural to write the solution of (C) as follows:

$$u(x, y) = H_n^{(\mu)}(F(x), y; q) \quad (3.8)$$

This last result completes the purposes of the present paper aimed at providing a general framework for the families of q -analogue exponential operators.



Particular case: By taking limit $q \mapsto 1$, the eqs. (3.1) to (3.8) reduce to the results due to Dattoli et al. [3; p. 220-221, section 3].

4. Concluding Remarks

In the previous section we have seen that the theory of q -exponential operators can be conveniently complemented by the use of functions satisfying recurrences of quasi monomial nature.

In these concluding remarks we will discuss the introduction of a family of functions which can be viewed as a fairly natural consequence of the so far developed formalism.

We consider the case of logarithmic Bessel functions, whose generating function can be cast in the following form

$$G(x, \vartheta; q) = x^{i \sin_q(\vartheta)} = \sum_{n=-\infty}^{\infty} e_q^{i n \vartheta} J_n(\ln_q(x); q), \quad (4.1)$$

where $J_n(x)$ denote the first kind cylinder Bessel functions, that is its q -analogue is written as

$$J_n(z; q) = \frac{(q^{n+1}; q)_{\infty}}{(q; q)_{\infty}} \left(\frac{1}{2}z\right)^n \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{4}z^2\right)^k}{(q; q)_k (q^{n+1}; q)_k}.$$

It is evident that we can take advantage from the discussion of the previous sections, to consider the following problem

$$e^{y \left(x \left[\frac{\partial}{\partial x}\right]_q\right)^2} x^{i \sin_q(\vartheta)} = e_q^{i \sin_q(\vartheta)} \left(\ln_q(x) + 2iy \left[\frac{\partial}{\partial x}\right]_q\right). \quad (4.2)$$

The q -exponential can be decoupled by means of the Weyl rule

$$e_q^{\hat{P} + \hat{Q}} = e_q^{\hat{P}} e_q^{\hat{Q}} e_q^{-\frac{1}{2}[\hat{P}, \hat{Q}]},$$

where

$$[\hat{P}, \hat{Q}] = \hat{P}\hat{Q} - \hat{Q}\hat{P},$$

by setting indeed

$$\left. \begin{aligned} \hat{P} &= i \sin_q(\vartheta) \ln_q(x), \\ \hat{Q} &= 2iy (\sin_q(\vartheta)) x \left[\frac{\partial}{\partial x}\right]_q, \end{aligned} \right\} \quad (4.3)$$

we find

$$[\hat{P}, \hat{Q}] = 2y [\sin_q(\vartheta)]^2, \quad (4.4)$$

thus getting

$$e_q^{y\left(x\left[\frac{\partial}{\partial x}\right]_q\right)^2} x^{i \sin_q(\vartheta)} = x^{i \sin_q(\vartheta)} e_q^{-y[\sin_q(\vartheta)]^2}. \quad (4.5)$$

Which is the generating function of a two-variable Bessel function, namely

$$x^{i \sin_q(\vartheta)} e_q^{-y[\sin_q(\vartheta)]^2} = \sum_{r=-\infty}^{\infty} e_q^{i r \vartheta} ({}_h J_r(x, y; q)). \quad (4.6)$$

$${}_h J_r(x, y; q) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{r+2r}(\ln_q(x), y; q)}{2^{r+2r} [r]_q! [n+r]_q!}. \quad (4.7)$$

It is evident that we ended up with a Bessel type function generalizing those of Hermite nature discussed in ref. [8]. It is worth emphasizing that the above equations satisfy a partial differential equation of the type

$$\begin{cases} \left[\frac{\partial}{\partial y} \right]_q ({}_h J_r(x, y; q)) = \left(x \left[\frac{\partial}{\partial x} \right]_q \right)^2 ({}_h J_r(x, y; q)), \\ {}_h J_r(x, 0; q) = J_r(\ln_q(x); q). \end{cases} \quad (4.8)$$

It is evident that the above considerations can be extended to any generating function of the type [3]

$$e^{iF(x) \sin_q(\vartheta)}. \quad (4.9)$$

Before concluding we will show how the combined use of integral transform and the previous formalism allows the derivation of further important relations. According to the identity [19]

$$e_q^{\lambda \hat{P}^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2 + 2\xi \sqrt{\lambda} \hat{P}} d_q \xi. \quad (4.10)$$

we can easily conclude that the polynomials $h_r^{(2)}(x, y; q)$ can also be realized in terms of the integral representation

$$e_q^{y\left(x\left[\frac{\partial}{\partial x}\right]_q\right)^2} [F(x)]^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} [F(x) + 2\sqrt{y}\xi]^n d_q \xi. \quad (4.11)$$

Going back to eq. (2.17) and specializing for $m = 2$, the use of the above relations allows to state the following identity

$$\sum_{n=0}^{\infty} \frac{(-1)^n e_q^{-yn^2}}{[n]_q!} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e_q^{-\xi^2} e_q^{-\infty s_q(2\sqrt{y}\xi)} \cos_q(\sin_q(2\sqrt{y}\xi)) d_q \xi.$$

Finally since [7]

$$e_q^{-yt} = \frac{y}{2\sqrt{\pi}} \int_0^{\infty} \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} e_q^{-d^2 t} d_q t \quad (4.12)$$

by replacing d with $\mathcal{T}_{a^+}^{\frac{1}{2}}$ and by using the previously discussed rules we find

$$e_q^{-y(\mathcal{T}_{a^+}^{\frac{1}{2}})} = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} e_q^{-(\mathcal{T}_{a^+}^{\frac{1}{2}})t} d_q t$$

let us see the effects of $e_q^{-y(\mathcal{T}_{a^+}^{\frac{1}{2}})}$ on an ordinary function, we have

$$e_q^{-y(\mathcal{T}_{a^+}^{\frac{1}{2}})} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} e_q^{-t(\mathcal{T}_{a^+}^{\frac{1}{2}})} f(x) d_q t$$

or

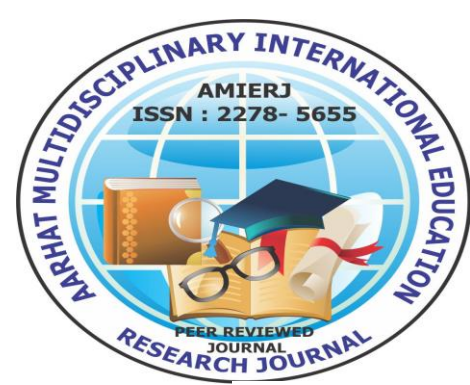
$$e_q^{-y(\mathcal{T}_{a^+}^{\frac{1}{2}})} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{1}{t\sqrt{t}} e_q^{-\frac{y^2}{4t}} f(F^{-1}(F(x) - t)) d_q t. \quad (4.13)$$

Which realizes the transform providing the action of a fractional generalized shift operator on a given function, the validity of both (3.11) and (3.12) is limited to the case in which the integral converges.

Particular case: Taking the limit $q \rightarrow 1$, the eqs. (4.1) to (4.13) give raise to the eqs. (15) to (26) due to Dattoli et al. [2].

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