ISSN–2277- 8721

EIIRJ Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

OPEN ACCESS Original Research Article

GENERAL SOLUTION OF LINEAR DIOPHANTINE EQUATION IN N UNKNOWNS AND APPLICATION OF LINEAR DIOPHANTINE EQUATION IN 2 UNKNOWNS OVER Zn

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Abstract:

In present paper, we study the most important concept in number theory "Linear Diophantine Equation in n Unknowns". Necessary and sufficient condition for the existence of solution and general the solution of linear Diophantine equation in n unknowns. We use the linear Diophantine equation in 2 unknowns in ring Zn to find the divisor, common divisor. We express a linear Diophantine equation over Zn in two unknowns and find the necessary and sufficient condition for the existence of solution of linear Diophantine equation and to find the solution over Zn.

Keywords: *Linear Diophantine equation, g.c.d(Greatest Common Divisor) , Zn (Congruence Ring).*

Introduction:

Consider a real life problem, a man wishes to purchases 500 Rs of 3 books, 4 notebooks and 10 pens. Then what is the price of a book, notebook, pen? To handle such problem, let x denotes the price of a book,y denotes the price of a notebook and z denotes the price of a pen. The given problem is converted into mathematical equation 3x+4y+10z=500. [1] To solve this problem, we need to find the all solution of the equation, where x, y and z are non-negative integers.

when we require that the integer solution of a particular equation, we have a Diophantine equation. An equation in one or more unknown to be solved with integer values are known as a Diophantine equation. Such Diophantine equation is initiated by the greatest Greek mathematician Diophantus of Alexandria. The simplest type of Diophantine equation is the linear Diophantine equation in two unknowns: $ax + by = c$,where a, b, c are integers and a, b are not both zero.This equation has a solution if and only if $d \mid c$, $d = gcd(a, b)$ .If x_0, y_0 is any particular solution this equation ,then general solution is given by

$x = x_0 \left(\frac{b}{d}\right) t$ and $y = y_0 - \left(\frac{a}{d}\right) t$,$t \in Z$ [1]. To solve the Diophantine equation in n unknowns defined in the definition, I have tried to find the formula for the general solution of

Diophantine equation in n unknowns.

General Solution of Linear Diophantine Equation in n Unknowns:

Definition 2.1. General form of linear Diophantine equation in n unknowns: - ([2]-page no. -67)A linear equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = c$ where $a_i, c \in Z, a_i \neq 0$,for all $1 \leq i \leq n$, is called a

ISSN–2277- 8721

Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

OPEN ACCESS Original Research Article

general form of linear Diophantine equation in n unknowns which has an integer solutions.

Proposition 2.2. Let $a_i \in Z, a_i \neq 0$,for all $1 \leq i \leq n$ and let $d = gcd(a_1, a_2, a_3, \dots a_n), d_2 = gcd(a_1, a_2), d_3 = gcd(d_2, a_3), d_4 = gcd(d_3, a_4) \dots d_n = gcd(d_n, a_{n-1}$.Then
$d = d_n$and $d = a_1t_1 + a_2t_2 + a_3t_3 + \cdots + a_nt_n$, for some $t_i \in Z$.

Proposition 2.3. ([2]) A general form of linear Diophantine equation $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = c$ where $a_i, c \in Z, a_i \neq 0$,for all $1 \leq i \leq n$ has a solution if and only $if\ d \mid c, d = gcd(a_1, a_1, a_1, \dots, a_n)$.

Theorem 2.4: -If $z_{1,}z_{2,}z_3 \cdots z_{n,}$ is any particular solution of the Linear Diophantine Equation$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = c$, where
$a_i \in Z, a_i \neq 0,$ for all $1 \leq i \leq n$, then all other solution is given by

$$x_j = z_j + \left(\frac{\prod_{\substack{i=1 \\ i\neq j}}^{n} a_{i,}}{d}\right) t, for\ all 1 \leq i \leq n-1, 1 \leq j \leq n-1, t \in Z,$$

$$x_n = z_n - \left(\frac{(n-1)\prod_{i=1}^{n-1} a_{i,}}{d}\right) t$$, where $d = gcd(a_1, a_2, a_3, \cdots a_n)$ such that $d \mid c$.

Proof: -Let $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = c$where$a_i \in Z, a_i \neq 0,$ for all $1 \leq i \leq n \dots (2.4.1)$ be a linearDiophantine Equation and let $d = gcd(a_1, a_2, a_3, \cdots, a_n)$ be such that $d \mid c$.Then by above theorem (2.3), linear Diophantine Equation has a solution.

Let $z_{1,}z_{2,}z_3 \cdots z_{n,}$ be any particular solution of the Linear Diophantine equation (2.4.1).

Then $a_1z_1 + a_2z_2 + a_3z_3 + \cdots + a_nz_n = c$ ... (2.4.2).

The matrix equation of equation (2.4.2) is given by

$A = XB \dots (2.4.3),\ A = [a_1\ \ a_2\ \ a_3\ \cdots\ a_n],\ X = [z_1, z_2, \dots, z_n]^T, B = [c]$

The augmented matrix is given by$[A{:}\,B] = [a_1\ \ a_2\ \ a_3\ \cdots\ a_n\ \ c\,]$.

Here $\rho(A) = 1$ and$\rho([A{:}\,B]) = 1$.

Thus , $\rho(A) = \rho([A{:}\,B]) = 1 < n =$ number of unknowns. Hence given system of equation (2.4.3) is consistent and has infinite number of solutions.

Therefore the system(2.4.3) has $n-1$ linearly independent solution (number of free variables). Without loss of generality we take $x_1, x_2, x_3, \cdots x_{n-1}$as $n-1$ non-zero free variables and is given by ,

$$x_j = z_j + \left(\frac{\prod_{\substack{i=1 \\ i\neq j}}^{n} a_{i,}}{d}\right) t, \text{ for all} 1 \leq i \leq n-1, 1 \leq j \leq n-1, t \in Z \dots(2.4.4)$$

To find variablex_n : -

Now consider

$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = c.$

$\therefore\ a_nx_n = c - (a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1},$
)

$$= c - \left\{a_1\left[z_1 + \left(\frac{\prod_{\substack{i=1 \\ i\neq 1}}^{n} a_{i,}}{d}\right)\right] + a_2\left[z_2 + \left(\frac{\prod_{\substack{i=1 \\ i\neq 2}}^{n} a_{i,}}{d}\right)t\right] \cdots + a_{n-1}\left[z_j + \left(\frac{\prod^{n}{}_{\substack{i=1 \\ i\neq n-1}}\ a_{i,}}{d}\right)t\right]\right\},$$

$$= c - \left\{(a_1z_1 + a_2z_2 + a_3z_3 + \cdots + a_{n-1}z_{n-1}) + \left((n-1)\frac{\prod_{i=1}^{n} a_{i,}}{d}\right)t\right\}, t \in Z,$$

$$= c - \left\{(c - a_nz_n) + \left((n-1)\frac{\prod_{i=1}^{n} a_{i,}}{d}\right)t\right\}, t \in Z,$$

ISSN–2277- 8721

EIIRJ Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

OPEN ACCESS Original Research Article

$$a_nx_n = c - c + a_nz_n - \left((n-1)\frac{\prod_{i=1}^{n} a_{i,}}{d}\right)t, t \in Z,$$

$$a_nx_n - a_nz_n = -\left((n-1)\frac{\prod_{i=1}^{n} a_{i,}}{d}\right)t, t \in Z,$$

$$a_n(x_n - z_n) = -\left((n-1)\frac{\prod_{i=1}^{n} a_{i,}}{d}\right)t, t \in Z,$$

$$(x_n - z_n) = -\left((n-1)\frac{\prod_{i=1}^{n} a_{i,}}{a_nd}\right)t, t \in Z,$$

$$\therefore x_n = z_n - \left((n-1)\frac{\prod_{\substack{i=1\\ i\neq j}}^{n-1} a_{i,}}{d}\right)t \text{, where } d = gcd(a_1, a_2, a_3, \cdots a_n) .$$

$$\therefore x_j = z_j + \left(\frac{\prod_{\substack{i=1\\ i\neq j}}^{n} a_{i,}}{d}\right)t, \text{ for all } 1 \le i \le n-1, 1 \le j \le n-1, t \in Z,$$

$$x_n = z_n - \left(\frac{(n-1)\prod_{\substack{i=1\\ i\neq j}}^{n-1} a_{i,}}{d}\right)t, d = gcd(a_1, a_2, a_3, \cdots a_n) \text{ such that } d \mid c \quad \cdots (2.4.5).$$

Thus, equation (2.4.5) gives the infinite number of linear Diophantine Equation (2.4.1).

We assumed that there exists $x_j = y_j \neq z_j + \left(\frac{\prod_{\substack{i=1\\ i\neq j}}^{n} a_{i,}}{d}\right)t,$

$t \in Z$or all $1 \le i \le n-1, \quad j \le n-1.$

Then $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = c,$

$$a_1 y_1 + a_2y_2 + a_3y_3 + \cdots + a_{n-1}y_{n-1} + a_ny_n = c,$$

$$a_ny_n = c - (a_1 y_1 + a_2y_2 + a_3y_3 + \cdots + a_{n-1}y_{n-1}),$$

$$\therefore y_n = \left(\frac{c}{a_n}\right) - \left(\frac{a_1}{a_n}\right)x_1 - \left(\frac{a_2}{a_n}\right)x_2 - \cdots \left(\frac{a_{n-1}}{a_n}\right)x_{n-1} = x_n \notin Z.$$

$\therefore x_1, x_2, x_3 + \cdots x_{n-1}, x_n$ is not integer solution of a general form of Linear Diophantine Equation in n −variables $x_1, x_2, x_3 + \cdots x_{n-1}, x_n$and is given by

$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = c$.

$$\therefore x_j = z_j + \left(\frac{\prod_{\substack{i=1\\ i\neq j}}^{n} a_{i,}}{d}\right)t, \text{ for all } 1 \le i \le n-1, 1 \le j \le n-1, t \in Z,$$

$x_n = z_n - \left(\frac{(n-1)\prod_{\substack{i=1\\ i\neq j}}^{n-1} a_{i,}}{d}\right)t, d = gcd(a_1, a_2, a_3, \cdots a_n), d \mid c$ is a general solution of linear Diophantine Equation (2.4.1).

Exercise 2.5: Solve the linear Diophantine equation$2x + 3y + 4z = 1$.

Solution: - Consider the linear Diophantine equation $2x + 3y + 4z = 1$.. (2.5.1).

Let $a_1 = 2, a_2 = 3, a_3 = 4, c = 1$.Here $d = gcd(a_1, a_2, a_3) = gcd(2,3,4) = 1$ and $d \mid c$, thus equation (2.5.1) has a solution. Let $x_0 = -1, y_0 = 1, z_0 = 0$ be a particular solution of equation (2.5.1) .Then the general solution is given by,

$$x = x_0 + \left(\frac{a_2a_3}{d}\right)t = -1 + (3.4)t = -1 + 12t,$$

$y = y_0 + \left(\frac{a_1 a_3}{d}\right) t = -1 + (2.4)t = -1 + 8t,$

$z = z_0 - 2\left(\frac{a_1 a_2}{d}\right) t = -2(2.3)t = -12t, t \in Z.$

$\therefore x = -1 + 12t, y = -1 + 8t, z = -12t, t \in Z$ is a general solution of the linear Diophantine equation $2x + 3y + 4z = 1.$

Exercise 2.6: Solve the linear Diophantine equation $2x_1 + 3x_2 - x_3 + 4x_4 = 5.$

Solution: - Consider the linear Diophantine equation

$2x_1 + 3x_2 - x_3 + 4x_4 = 5 \ldots (2.6.1).$

Let t$a_1 = 2, a_2 = 3, a_3 = -1, a_4 = 4, c = 5.$

Here $d = gcd(a_1, a_2, a_3, a_4) = gcd(2,3,-1,4) = 1$ and $d \mid c$, thus equation $(2.6.1)$ has a solution. Let $z_1 = 1, z_2 = 1, z_3 = 0, z_4 = 0$ be a particular solution of equation $(2.61.)$ .Then the general solution is given by,

$x_1 = z_1 + \left(\frac{a_2 a_3\ a_4}{d}\right) t = 1 + (3. -4)t = 1 - 12t,$

$x_2 = z_2 + \left(\frac{a_1 a_3\ a_4}{d}\right) t = 1 + (2. -4)t = 1 - 8t,$

$x_3 = z_3 + \left(\frac{a_1 a_2 a_4}{d}\right) t = 0 - (2.3.4)t = -24t,$

$x_4 = z_4 - 3\left(\frac{a_1 a_2 a_3}{d}\right) t = 0 - (2.3. -1)t = 6t, t \in Z.$

$\therefore x_1 = 1 - 12t, x_2 = 1 - 8t, x_3 = -24t, x_4 = 6t, t \in Z$ is a general solution of the linear Diophantine equation $2x_1 + 3x_2 - x_3 + 4x_4 = 5.$

1) Solve the linear Diophantine equation

$2x_1 + x_2 + x_3 + 4x_4 + 2x_5 + 4x_6 - 3x_7 + x_8 - 3x_9 + x_{10} = 1.$

Solution: -

Consider the linear Diophantine equation

$2x_1 + x_2 + x_3 + 4x_4 + 2x_5 + 4x_6 - 3x_7 + x_8 - 3x_9 + x_{10} = 1 \ldots (a).$

Compare equation (I) with the linear Diophantine equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 + a_6x_6 + a_7x_7 + a_8x_8 + a_9x_9 + a_{10}x_{10} = c$, we get$a_1 = 2, a_2 = 1, a_3 = 1, a_4 = 4\ , a_5 = 2, a_6 = 4, a_7 = -3, a_8 = 1, a_9 = -3, a_{10} = 1, c = 1.$Here $d = gcd(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) = 1$ and $d \mid c$, thus equation (I) has a solution. Let $z_1 = 1, z_2 = -1, z_3 = 0, z_4 = 0, z_5 = 0, z_6 = 0\ , z_7 = 0, z_8 = 0, z_9 = 0, z_{10} = 0$ be a particular solution of equation (a).Then the general solution is given by,

$x_1 = z_1 + \left(\frac{a_2 a_3\ a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}{d}\right) t = 1 + 288t,$

$x_2 = z_2 + \left(\frac{a_1 a_3\ a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}{d}\right) t = -1 + 576t,$

$x_3 = z_3 + \left(\frac{a_1 a_2 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}}{d}\right) t = 0 + 576t = 576t,$

$x_4 = z_4 + \left(\frac{a_1 a_2 a_3 a_5 a_6 a_7 a_8 a_9 a_{10}}{d}\right) t = 0 + 144t = 144t,$

$x_5 = z_5 + \left(\frac{a_1 a_2 a_3 a_4 a_6 a_7 a_8 a_9 a_{10}}{d}\right) t = 0 + 288t = 288t,$

$x_6 = z_6 + \left(\frac{a_1 a_2 a_3 a_4 a_5 a_7 a_8 a_9 a_{10}}{d}\right) t = 0 + 144t = 144t,$

$x_7 = z_7 + \left(\frac{a_1 a_2 a_3 a_4 a_5 a_6 a_8 a_9 a_{10}}{d}\right) t = 0 - 192t = -192t,$

ISSN–2277- 8721

Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

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$x_8 = z_8 + \left(\frac{a_1a_2a_3a_4a_5a_6a_7a_9a_{10}}{d}\right)t = 0 + 576t = 576t,$

$x_9 = z_9 + \left(\frac{a_1a_2a_3a_4a_5a_6a_7a_8a_{10}}{d}\right)t = 0 - 192t = -192t,$

$x_{10} = z_{10} - 9\left(\frac{a_1a_2a_3\, a_4a_5a_6a_7a_8a_9}{d}\right)t = 0 - 5184t = -5184t, t \in Z.$

$\therefore x_1 = 1 + 228t, x_2 = -1 + 576t, x_3 = 576t, x_4 = 144t, x_5 = 288t, x_6 = 144t,$

$x_7 = -192t, x_8 = 576t, x_9 = -192t, x_{10} = -5184t, t \in Z$ is a general solution of the linear Diophantine equation $2x_1 + x_2 + x_3 + 4x_4 + 2x_5 + 4x_6 - 3x_7 + x_8 - 3x_9 + x_{10} = 1.$

Application of Linear Diophantine Equation in Z_n:-

A) To find the common divisors over Z_n.

Theorem3.1: -Let $\bar{a}, \bar{b} \in Z_n, \bar{a} \neq \bar{0}$.Then $\bar{a} \mid \bar{b}$ if and only if $d \mid b$ where $d = gcd(a, n)$.

Proof: -Let $\bar{a}, \bar{b} \in Z_n, \bar{a} \neq \bar{0}$ .Then

$\bar{a} = a + \langle n \rangle = \{a + nt : t \in Z\}$ and $\bar{b} = b + \langle n \rangle = \{b + nt : t \in Z\}$

Let $\bar{a} \mid \bar{b}$ .Then there exist $\bar{x} \in Z_n, \bar{x} \neq 0$ such that $\bar{b} = \bar{a} \times_n \bar{x}$.

$b + \langle n \rangle = (a + \langle n \rangle)(x + \langle n \rangle) = (ax) \langle n \rangle$

For modulo n, $b \equiv ax (mod\ n)$, for some $0 \leq x < n \cdots (3.1.1)$

Thus, $n \mid ax - b$, for some $0 \leq x < n$

There exists $y \in Z$ such that $ax - b = ny$, for some $0 \leq x < n$.

$\therefore ax + ny = b \cdots ((3.1.2))$.

Which is a linear Diophantine equation in x and y .Hence equation (3.1.2), has a solution if and only one $d \mid b$ where $d = gcd(a, n)$.Thus $\bar{a} \mid \bar{b}$ if and only if $d \mid b, d = gcd(a, n)$.

Corollary3.2. In Z_p, p is a prime number, $\bar{a} \mid \bar{b}$,for any $\bar{a}, \bar{b} \in Z_p, \bar{a} \neq \bar{0}$.

Corollary 3.3: Let $\bar{a}, \bar{b}, \bar{c} \in Z_n, \bar{c} \neq \bar{0}$ .Then an element $\bar{c}$ is called a common divisor of $\bar{a}$and $\bar{b}$ if and only if d is a common divisor of a and b, $d = gcd(c, n)$.

Theorem 3.4: -Let $a, b \in N$.If $d = gcd(a, b)$ then $\bar{d} \mid \bar{a}$ and $\bar{d} \mid \bar{b}$ in Z_n, for any $max\{a, b, d\} < n$.

Proof: -Let $a, b \in N$ be such that $d = gcd(a, b)$.Then $d \mid a$ and $d \mid b$.Thus $m, k \in N$ Such that $a = dm$ and $b = dk$ where $m < a$ and $k < b$.We choose $n \in N$ such

That $max\{a, b, d\} < n$, clearly $m < n$ and $k < n$ .Let $a = dm$ .Then $a + nt = dm + nt$, $t \in Z$.Thus $a + \langle n \rangle = dm + \langle n \rangle, n \in N$ and $m, d < n$,

$a + \langle n \rangle = = (d + \langle n \rangle) \times_n (m + \langle n \rangle), n \in N$

Thus $\bar{a} = \bar{d} \times_n \bar{m}$, $max\{a, b, d\} < n$.Similarly, we can show that$\bar{b} = \bar{d} \times_n \bar{k}$.

Therefore $\bar{d} \mid \bar{a}$ and $\bar{d} \mid \bar{b}$ in Z_n, for any $max\{a, b, d\} < n$ .Hence $\bar{d}$ is a common divisor of $\bar{a}$and $\bar{b}$ in Z_n, for any $max\{a, b, d\} < n$.

Exercise 3.5 Find all divisor of $\bar{2}$ inZ_8.

Solution: -Let$Z_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ and $d =$ gcd $(a, 8)$ such that $d \mid 2$.By theorem (3.1),d $\mid 2 \Rightarrow$d=1, 2.

For $d = 1, \bar{a} = \bar{1}, \bar{3}\ , \bar{5}\ , \bar{7}$.

And for $d = 2, \bar{a} = \bar{2}\, \bar{4}, , \bar{6}$.

Thus , $\bar{a} = \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}$ are divisors of $\bar{2}$ in Z_8 .

ISSN–2277- 8721

EIIRJ Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

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Exercise 3.6 Find a common divisor of $\bar{3}$ and $\bar{4}$ in Z_6.

Solution: -Let$Z_6 = \{\bar{0}, \bar{1}, \bar{2}\,\bar{3}\,\bar{4}, \bar{5}\}$ and $d = \gcd(a, 6)$ such that $d\ |3, d\ |4$. . By theorem (3.1) and corollary (3.3), d is a common divisor of 3 and 4. Thus d=1.For $d = 1, \bar{a} = \bar{1}$, $\bar{5}$ in Z_6.Thus $\bar{1}$, $\bar{5}$ are a common divisor of $\bar{3}$ and $\bar{4}$ in Z_6.

B)Linear Diophantine Equation overZ_n:

Definition 3.7.A linear equation$\bar{a}x + \bar{b}y = \bar{c}$, where $\bar{a}, \bar{b}\bar{c}, \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$, which has a solution in Z_n i.e., has an integer modulo-n solution is called as Linear Diophantine Equation overZ_n.Here we replace$\bar{x}, \bar{y}\ by\ x, y\ and\ +_n, \times_n$by + and . (dot) respectively.

Theorem 3.8: -A linear Diophantine equation$\bar{a}x + \bar{b}y = \bar{c}, \bar{a}, \bar{b}, \bar{c}, \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$ has a solution if and only if $d\ |c$, where $d = gcd(a, b, n)$.

Proof: - Let $\bar{a}, \bar{b}, \bar{c} \in Z_n$ .Then $\bar{a} = a + < n >, \bar{b} = b + < n >\ and\ \bar{c} = c + < n >$

Let $\bar{a}x + \bar{b}y = \bar{c}$, where $\bar{a}, \bar{b}, \bar{c} \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0} \cdots$ (3.8.1)be a linear Diophantine equation over Z_n .Then $(a + < n >)x + (a + < n >)y = (c + < n >)$

For modulo n, we get $ax + by \equiv c(mod\ n)$, hence $n\ |(ax + by) - c$.Therefore there exists $z \in Z$ Such that $(ax + by) - c = nz$, hence $ax + by + nz = c \cdots$ (3.8.2). is a linear Diophantine equation in three variables x, y and z. Equation (3.8.2) has a solution if and only if $d\ |c, d = gcd(a, b, n)$.Therefor a linear Diophantine equation $\bar{a}x + \bar{b}y = \bar{c}, \bar{a}, \bar{b}, \bar{c} \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$has a solution if and only if $d\ |c, d = gcd(a, b, n)$.

Corollary 3.:9 A linear Diophantine equation$\bar{a}x + \bar{b}y = \bar{c}$, where $\bar{a}, \bar{b}, \bar{c}, \in Z_p, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}, p$ is a prime number has a always solution in a field Z_p .

Theorem3.10: -Let$\bar{a}x + \bar{b}y = \bar{c}$, where $\bar{a}, \bar{b}, \bar{c} \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$a linear Diophantine equation be. Suppose that $d\ |c$, where $d = gcd(a, b, n)$and $e\ |c$, where $e = gcd(a, b)$.

If x_0,y_0 is any particular solution of a linear Diophantine equation $ax + by = c$ where $a, b, c \in Z, a \neq 0, b \neq 0$ .Then $x = \bar{r}, y = \bar{s}$ is a solution of a linear Diophantine equation over Z_n, where

$$x = x_0 + \left(\frac{b}{d}\right)t \equiv r(mod\ n),$$

$$y = y_0 - \left(\frac{a}{d}\right)t \equiv s(mod\ n), t \in Z.$$

Proof: -Let$\bar{a}x + \bar{b}y = \bar{c}$, $\bar{a}, \bar{b}, \bar{c}, \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0} \cdots$ (3.10.1) be a linear Diophantine equation overZ_n .We know that for any $a, b, c \in Z, a \in \bar{a} = [a], b \in \bar{b} = [b], c \in \bar{c} = [c]$.

Therefore equation (3.10.1) can be transferred into linear Diophantine equation over Z and is given by $ax + by = c, a, b, c \in Z, a \neq 0, b \neq 0 \cdots$ (3.10.2).

Let $e\ |c$, where $e = gcd(a, b)$.Thus equation (3.10.2) has a solution. Let x_0, y_0be any particular solution of a linear Diophantine equation(3.10.2).Then the general solution of equation (3.10.2) is given by

$$x = x_0 + \left(\frac{b}{e}\right)t\ and\ y = y_0 - \left(\frac{a}{e}\right)t, t \in Z.$$

To obtain the solution of equation (3.10.1): $-$

$$x = x_0 + \left(\frac{b}{e}\right)t \equiv r(mod\ n), y = y_0 - \left(\frac{a}{e}\right)t \equiv s(mod\ n), t \in Z, \text{where } 0 \leq r, s < n.$$

Clearly $r, s \in Z_n$.

We show that $x = r, y = s$ is a solution of equation 3.10.1): $-$

$x \equiv r(mod\ n), y \equiv s(mod\ n)$

$ax \equiv ar\ (mod\ n) and\ by \equiv bs(mod\ n)\ , since\ a \neq 0, b \neq 0$, we get

$ar + bs \equiv c(mod\ n), 0 \leq r, s < n$. Hence $x = \bar{r}, y = \bar{s}$ is a solution of a linear Diophantine equation over Z_n, where $x = x_0 + \left(\frac{b}{e}\right) t \equiv r(mod\ n), y = y_0 - \left(\frac{a}{e}\right) t \equiv s(mod\ n), t \in Z$.

Theorem 3.11: -Let $\bar{a}x + \bar{b}y = \bar{c}$, where $\bar{a}, \bar{b}, \bar{c} \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$ a linear Diophantine equation be. Suppose that $d \mid c$, where $d = gcd(a, b, n)$and $e \mid c$, where $e = gcd(a, b)$.

Then the linear Diophantine equation $\bar{a}x + \bar{b}y = \bar{c}$ has exactly distinct n solution in Z_n .

Proof: - Let $\bar{a}x + \bar{b}y = \bar{c}$, where $\bar{a}, \bar{b}, \bar{c} \in Z_n, \bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$ a linear Diophantine equation be. Suppose that $d \mid c$, where $d = gcd(a, b, n)$and $e \mid c$, where $e = gcd(a, b)$.

Then by above theorem (3.10),ifx_0,y_0 is any particular solution of a linear Diophantine equation $ax + by = c$ where $a, b, c \in Z, a \neq 0, b \neq 0$ .Then $x = \bar{r}, y = \bar{s}$ is a solution of a linear Diophantine equation over Z_n , where

$x = x_0 + \left(\frac{b}{e}\right) t \equiv r(mod\ n) y = y_0 - \left(\frac{a}{e}\right) t \equiv s(mod\ n), t \in Z$ where $0 \leq r, s < n$ ... (3.11.1)

Case (I) If $0 \leq t < n$, then by equation (3.11.1), we get

For t=0, we get,

$x = x_0 \equiv r_0(mod\ n), y = y_0 \equiv s_0(mod\ n)$.

For $t = 1$, we get ,

$x = x_0 + \left(\frac{b}{e}\right) \equiv r_1(mod\ n),\ y = y_0 - \left(\frac{a}{e}\right) \equiv s_1(mod\ n)$.

For , $t = 2$, we get ,

$x = x_0 + 2\left(\frac{b}{e}\right) \equiv r_2(mod\ n) , y = y_0 - 2\left(\frac{a}{e}\right) \equiv s_2(mod\ n)$.

.. For $t = n - 1$, we get,

$x = x_0 + (n-1)\left(\frac{b}{e}\right) \equiv r_{n-1}(mod\ n), y = y_0 - (n-1)\left(\frac{a}{e}\right) \equiv s_{n-1}(mod\ n)$.

Thus for $0 \leq t < n$, we $n -$ distinct solutions$(r_0, s_0), (r_1, s_1), \ldots (r_{n-1}, s_{n-1})$ of linear Diophantine equation $\bar{a}x + \bar{b}y = \bar{c}$ in Z_n .

Case (II) If $t > n\ and\ t < n$ then by division algorithm, there exists unique integer pair ∴ (p, q) such that $t = nq + p$, where $0 \leq p < n$

∴ $t \equiv p(modn)$, where $0 \leq p < n$ ... (3.11.2)

By case ($\boldsymbol{I}$) and equation (11.2), we conclude that for ant $t > n, t < n$,we get one of the pair $(r_i, s_i), 0 \leq i \leq n - 1$

∴By case ($\boldsymbol{I}$) and ($\boldsymbol{II}$), we conclude that the linear Diophantine equation $\bar{a}x + \bar{b}y = \bar{c}$ has exactly distinct n solution in Z_n .

Exercise 3.12 Find of the solutions linear Diophantine equation $\bar{2}x + \bar{3}y = \bar{7}$ in Z_8.

Solution: Consider the linear Diophantine equation $\bar{2}x + \bar{3}y = \bar{7}$ ... (3.12.1)

Here $a = 2, b = 3, c = 7, n = 8$ and $d = \gcd(a, b, n) = \gcd(2,3,8) = 1, e = \gcd(a, b) = \gcd(2,3) = 1$ also $d|c$ and $e|c$ .By theorem (3.8) , equation (3.12.1) has a solution in Z_8

The linear Diophantine equation over Z and is given by $2x + 3y = 7$ ... (3.12.2)

ISSN–2277- 8721

Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

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Since $e|c$, where $e = \gcd(a, b) = \gcd(2,3) = 1$, thus equation (3.12.2) has a solution in Z.Let $x_0 = 2, y_0 = 1$ be a particular solution of equation (3.12.2) .Then the general solution is given by

$x = x_0 + \left(\frac{b}{e}\right)t = 2 + \left(\frac{3}{1}\right)t = 2 + 3t \ldots$ (3.12.3)

$y = y_0 - \left(\frac{a}{e}\right)t = 2 - \left(\frac{2}{1}\right)t = 1 - 2t, t \in Z$ ... (3.12.4)

By putting $t = 0,1,2,3,4,5,6,7$ in equations (3.12.3) and (.12.4), we get

(I) For $t = 0$,

$x = 2 \equiv 2(mod8), y = 1 \equiv 1(mod8) \Rightarrow r_0 = \bar{2}, s_0 = \bar{1}$.

(II)For $t = 1$

$x = 5 \equiv 5(mod8), y = -1 \equiv 7(mod8) \Rightarrow r_1 = \bar{5}, s_1 = \bar{7}$.

(III) For $t = 2$

$x = 8 \equiv 0(mod8), y = -3 \equiv 5(mod8) \Rightarrow r_2 = \bar{0}, s_2 = \bar{5}$.

(IV)For $t = 3$

$x = 11 \equiv 3(mod8), y = -5 \equiv 3(mod8) \Rightarrow r_3 = \bar{3}, s_3 = \bar{3}$.

(V) For $t = 4$

$x = 14 \equiv 6(mod8), y = -7 \equiv 1(mod8) \Rightarrow r_4 = \bar{6}, s_4 = \bar{1}$.

(VI) For $t = 5$

$x = 17 \equiv 1(mod8), y = -9 \equiv 7(mod8) \Rightarrow r_5 = \bar{1}, s_5 = \bar{7}$.

(VII) For $t = 6$

$x = 20 \equiv 4(mod8), y = -11 \equiv 5(mod8) \Rightarrow r_6 = \bar{4}, s_6 = \bar{5}$.

(VIII) For $t = 7$

$x = 23 \equiv 7(mod8), y = -13 \equiv 3(mod8) \Rightarrow r_7 = \bar{7}, s_7 = \bar{3}$.

Hence $(\bar{2},\bar{1}),(\bar{5},\bar{7}),(\bar{0},\bar{5}),(\bar{3},\bar{3})(\bar{6},\bar{1}),(\bar{1},\bar{7}),(\bar{4},\bar{6}),(\bar{7},\bar{3})$ are the 8 −distinct solutions of linear Diophantine equation $\bar{2}x + \bar{3}y = \bar{7}$ in Z_8.

Exercise 3.13: Show that the linear Diophantine equation $\bar{2}x + \bar{4}y = \bar{7}$ has no solution in Z_8.

Solution: Consider the linear Diophantine equation, $\bar{2}x + 4y = \bar{7}$ ... (3.13.1).

Here$a = 2, b = 4, c = 7, n = 8$,$d = \gcd(a, b, n) = \gcd(2,4,8) = 2, e = \gcd(a, b) = \gcd(2,4) = 2$.Here$e \nmid c$.Therefore by theorem (3.8), equation (3.14.1) has no solution in Z_8

Exercise3.14. Find the solutions of linear Diophantine equation $\bar{2}x - \bar{3}y = \bar{4}$ in Z_5.

Solution : Consider the linear Diophantine equation$\bar{2}x - \bar{3}y = \bar{4}$ ... (3.14.1).

The additive inverse of $\bar{3}$ in Z_5 is $\bar{2}$, thus the equation (3.14.1) can be written as

$\bar{2}x + \bar{2}y = \bar{4}$ ... (3.14.2).

Here$a = 2, b = 2, c = 4, n = 5.d = \gcd(a, b, n) = \gcd(2,2,5) = 1$ and $e = \gcd(a, b) = \gcd(2,2) = 2$.Here $d|c$ and$e|c$.Equation (3.14.2) has a solution in Z_5 .

Consider the linear Diophantine equation over Z and is given by,

$2x + 2y = 4$ ... (3.14.3).Since $e|c$, where $e = \gcd(a, b) = \gcd(2,2) = 2$, thus equation (3.14.3) has a solution in Z.Let $x_0 = 2, y_0 = 2$ be a particular solution of equation (3.14.3) .Then the general solution is given by

$x = x_0 + \left(\frac{b}{e}\right)t = 2 + \left(\frac{2}{2}\right)t = 2 + t \ldots$ (3.14.4)

ISSN–2277- 8721

EIIRJ Electronic International Interdisciplinary Research Journal

Volume–XII, Issues – VI(Special Issues-I) Nov – Dec 2023

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$y = y_0 - \left(\frac{a}{e}\right)t = 2 - \left(\frac{2}{2}\right)t = 2 - t, t \in Z$ … (3.14.5)

By putting $t = 0,1,2,3,4$ in equations (3.14.4) and(3.14.5), we get

(I) For $t = 0$,

$x = 2 \equiv 2(mod5), y = 2 \equiv 2(mod5) \Rightarrow r_0 = \bar{2}, s_0 = \bar{2}$.

(II)For $t = 1$

$x = 3 \equiv 3(mod5), y = 1 \equiv 1(mod5) \Rightarrow r_1 = \bar{3}, s_1 = \bar{1}$.

(III) For $t = 2$

$x = 4 \equiv 4(mod5), y = 0 \equiv 0(mod5) \Rightarrow r_2 = \bar{4}, s_2 = \bar{0}$.

(IV)For $t = 3$

$x = 5 \equiv 0(mod5), y = -1 \equiv 4(mod5) \Rightarrow r_3 = \bar{0}, s_3 = \bar{4}$.

(V) For$t = 4$,

$x = 6 \equiv 1(mod5), y = -2 \equiv 3(mod5) \Rightarrow r_4 = \bar{1}, s_4 = \bar{3}$.

Therefore $(\bar{2}, \bar{2}), (\bar{3}, \bar{1}), (\bar{4}, \bar{0}), (\bar{0}, \bar{4}), (\bar{1}, \bar{3})$are the 5 $-$distinct solutions of linear Diophantine equation $\bar{2}x - \bar{3}y = \bar{4}$ in Z_5.

Conclusion:

In this way, we study the general solution of linear Diophantine equation in n unknowns and used linear Diophantine equation in two variables over Z_n to find the common divisor and defined the linear Diophantine equation over Z_n

Acknowledgement:

Thanks to all respected teachers to inspire me for research in Mathematics and thanks to all reviewers for carefully reading the manuscript and for the kind comments for improvement of the paper.

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Cite This Article:

Mr. Kumbhar S.S. (2023). GENERAL SOLUTION OF LINEAR DIOPHANTINE EQUATION IN N UNKNOWNS AND APPLICATION OF LINEAR DIOPHANTINE EQUATION IN 2 UNKNOWNS OVER ZN. In Electronic International Interdisciplinary Research Journal: Vol. XII (Number VI, pp. 100–109). Zenodo. https://doi.org/10.5281/zenodo.10461129